Name ______________________________ **Date** ____________

An Extreme Value Problem

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CALCULATORS: *Casio ClassPad 300*

Student Exercises

EXERCISE 1

Let $f(x) = xe^{-x}$. Use your *ClassPad* or direct computation to show that $f'(x) = e^{-x}(1 - x)$. Use the derivative of f to explain why the maximum value of f on the interval $[0, \infty)$ must occur at $x = 1$. [See ANSWERS for the derivation and explanation.] Thus, the maximum value of f on $[0, \infty)$ is $1/e$.

EXERCISE 2

Let $f(x) = xe^{-x/2}$. Show that the maximum value of f on $[0, \infty)$ is $2/e$.

EXERCISE 3

Let $f(x) = xe^{-2x}$. Show that the maximum value of f on $[0, \infty)$ is $1/(2e)$.

EXERCISE 4

Now let $f_a(x) = xe^{-ax}$ where a is a positive parameter. You worked with the cases a equal 1, 1/2 and 2 respectively in the previous three exercises. For each value of a, the function $f_a(x)$ has a maximum value on the interval $[0, \infty)$. You found these maximum values in special cases in the previous three exercises. Thus, we can define a function, which we call $F(a)$, for positive a, as follows:

$F(a)$ is the maximum value of $f_a(x) = xe^{-ax}$ on the interval $[0, \infty)$.

Find, if they exist, the maximum and the minimum values of $F(a)$ for $0 < a < \infty$.

EXERCISE 5

In Exercise 4, you should have discovered that $F(a) = \frac{1}{ae}$. Use the substitution $x = t/a$ in $f_a(x) = xe^{-ax}$.

Then use your answer to Exercise 1 to verify that the value of $F(a)$ really is $\frac{1}{ae}$.

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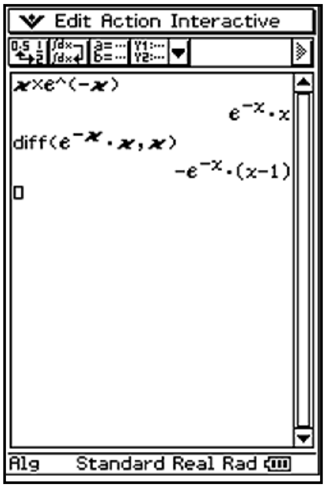
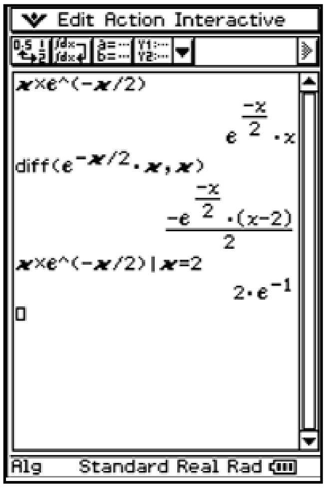
Answers

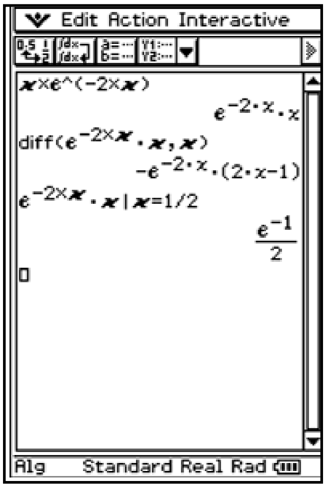
1. The screen capture at the left below shows that $f'(x) = e^{-x}(1 - x)$. It follows that the derivative is positive if $x < 1$ and the derivative is negative if $x > 1$. It follows that f has its maximum at $x = 1$.

2. The screen capture in the middle below shows the derivative of f goes from positive to negative at $x = 2$.

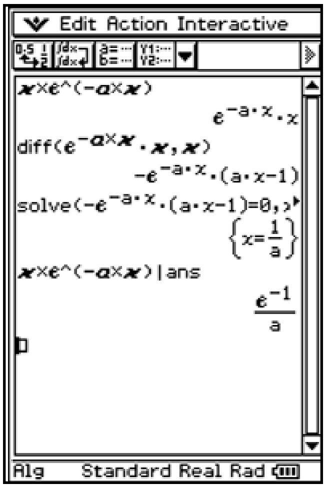
3. The screen capture at the right below shows the derivative of f goes from positive to negative at $x = 1/2$.







4. The screen capture at the right shows that $f'_a(x) = e^{-ax}(1 - ax)$ which goes from positive to negative at $x = 1/a$. We also see from the screen capture that $f_a(1/a) = \frac{1}{ae}$, so we have $F(a) = \frac{1}{ae}$. Finally, we note that this function of a has no maximum and no minimum for a such that $0 < a < \infty$.



5. $f_a(t/a) = \frac{te^{-t}}{a}$ and from Exercise 1, we know that the maximum of te^{-t} for positive t is $1/e$.
Thus, the maximum value of $f_a(x) = \frac{1}{ae}$. I.e. $F(a) = \frac{1}{ae}$.

