Name ______________________________ Date ____________

An Improper Integral

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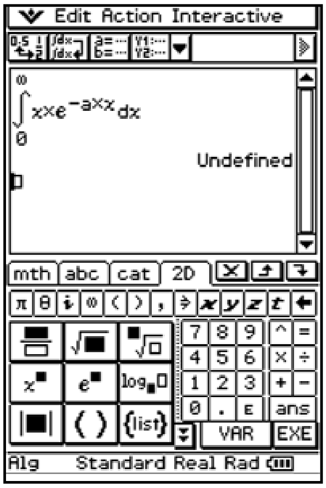
CALCULATORS: *Casio ClassPad 300*

Student Exercises

EXERCISE 1

Consider the improper integral: $\int_0^\infty xe^{-ax}dx$ where a is a positive real number.

We will see that this improper integral does converge, but the screen capture below shows that we cannot determine the value of the improper integral directly. The problem is that the *ClassPad* does not "know" that a is positive, and, if a were 0 or negative, the improper integral would diverge.



EXERCISE 1

Use your *ClassPad* to determine the value of the improper integral for a =1.

EXERCISE 2

Use your *ClassPad* to determine the value of the improper integral for a =3 and 5 and use your answers to make a guess at the value of improper integral for an arbitrary a.

If we are going to evaluate the improper integral for arbitrary a, we will need to do so indirectly. The following exercise provides one approach.

EXERCISE 3

Use your *ClassPad* to find an antiderivative for xe^{-ax} , and let $F_a(x)$ denote the antiderivative. Then the value of the improper integral is given by:

$$\left(\lim_{x\to\infty} F_a(x)\right) - F_a(0)$$

You will find that your *ClassPad* cannot evaluate the limit in this expression. You will need to use l'Hopital's Rule.

EXERCISE 4

Verify your answer to Exercise 3 by using the u-substitution $u = ax$ in the improper integral. You can use your answer to Exercise 1 to obtain the value of the original integral.

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continued

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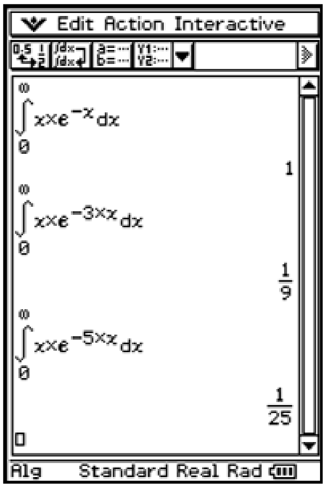
Answers

EXERCISES 1 & 2

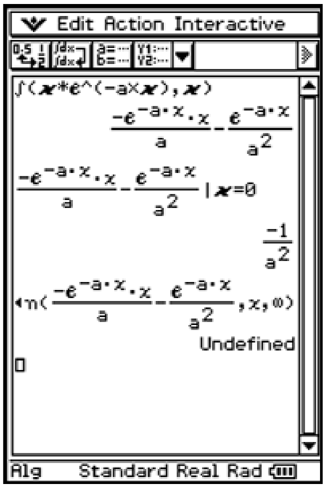


The screen capture at the right shows that $\int_0^\infty xe^{-ax}dx$ equals 1, 1/9 and 1/25 when a = 1, 3 and 5, respectively.

Thus, we guess that in general $\int_0^\infty xe^{-ax}dx = \frac{1}{a^2}$.

You might try a value of a that is less than 1 to obtain additional evidence.

EXERCISE 3



The screen capture at the right shows that the antiderivative $F_a(x)$ is $\frac{-e^{-ax}(ax+1)}{a^2}$ and that the value of the antiderivative at $x = 0$ is $\frac{-1}{a^2}$.

We also see that we cannot take the limit of $F_a(x)$ as x goes to infinity directly. However, we can use l'Hopital's Rule if we write $F_a(x) = \frac{ax+1}{-a^2e^{ax}}$.

It follows immediately that the value of $\int_0^\infty xe^{-ax}dx$ is:

$$\left(\lim_{x\to\infty} F_a(x)\right) - F_a(0) = 0 - \left(\frac{-1}{a^2}\right) = \frac{1}{a^2} .$$

[Again, the *ClassPad* cannot compute the limit because it does not "know" that a is positive.]

EXERCISE 4

With $u = ax$, $dx = \frac{1}{a}\,du$ and we obtain $\int_0^\infty xe^{-ax}dx = \int_{0=u}^\infty \frac{u}{a}\,e^{-u}\frac{1}{a}\,du = \frac{1}{a^2}\int_{0=u}^\infty ue^{-u}du$.

Since we know from Exercise 1 that $\int_{0=u}^\infty ue^{-u}du = 1$, we see that $\int_0^\infty xe^{-ax}dx = \frac{1}{a^2}$ when a >0.

