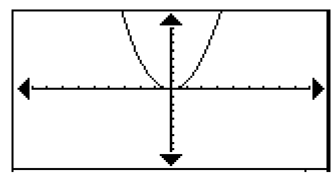
Understanding the Definition of a Limit
Using a ClassPad 300

C.C. Edwards
Coastal Carolina University

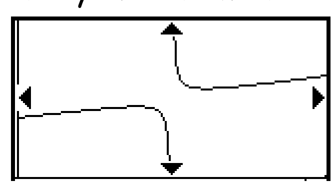
Before taking on the challenge of the "epsilon- delta" definition of a limit, let's look at an example which, at first, may not appear to have anything to do with the formal, "epsilon- delta" definition of a limit.

EXAMPLE 1. Describe the graph of $f(x) = \dfrac{x^3 - 10x^2 + 1}{x - 10}$.

Use the W application to graph the function. Since you don't know what the graph looks like, you would most likely tap 6 **Memory Standard** to graph the function in the standard viewing window.



But then you would probably realize that the really interesting part of the graph is around $x = 10$ where the function is undefined. So you would press 6 and set the *x* values between 9 and 11. Since you don't want to take the time to figure out what the corresponding *y* values should be, you would press **Zoom Auto** to get the ClassPad to figure them out for you and sketch the graph.



To recap: In graphing this function, ***you*** set the *x*-values and let the ClassPad figure out the corresponding *y*-values for you by using **Zoom Auto**.

What does this have to do with the definition of a limit? Let's look at that definition to answer this question.

The definition of a limit states:

Let $f(x)$ be defined for all values of *x* in ***some*** open interval containing the number *a*, with the possible exception that $f(x)$ need not be defined at *a*. Then $\lim_{x \to a} f(x) = L$ if ***given*** any number $\varepsilon > 0$, ***you***

can find a number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

Let's rephrase this definition.

To prove that $\lim_{x \to a} f(x) = L$, you must:

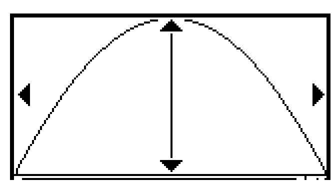
a. Find ***some*** open interval containing the number *a* and having the property that $f(x)$ is defined for all *x* in this interval with the exception that $f(x)$ does not have to be defined at *x* = *a*.

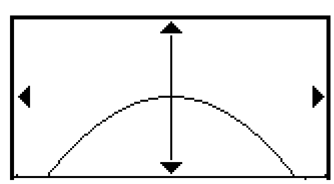
b. Show that if you are given any $\varepsilon > 0$ such that $L - \varepsilon < f(x) < L + \varepsilon$, then you can find a number $\delta > 0$ such that $a - \delta < x < a + \delta$.

Students rarely have trouble with part a of the definition, but they do have trouble understanding part b. This is where the calculator helps.

In part b of this definition the *y*-values are set by the choice of ε and ***you have to find*** the value of δ which gives the corresponding *x*-values. This is the opposite of what we did in example 1 when we used **Zoom Auto**! Unfortunately, the ClassPad does not have a feature like **Zoom Auto** which automatically determined the *x* values when you specify the *y* values, but the ClassPad can still be used to help you find these *x* values, as is illustrated in the following example.

EXAMPLE 2. $\lim_{x \to 0} \frac{\sin x}{x}$.

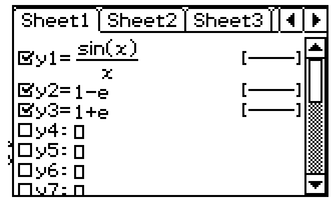




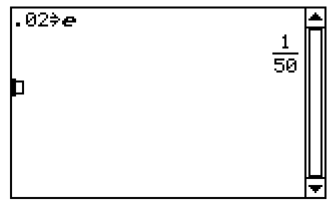
In the first graph at the right, $f(x) = \frac{\sin x}{x}$ is graphed using **Zoom Auto** with $-1 \le x \le 1$. (Why did we choose these *x* values?) What does not appear in the graph is the hole at $x = 0$. (Why?) Using **Analysis Trace** indicates that the limit might be 1, so in the second graph the *y* values are changed to $0.9 \le y \le 1.1$.

We are now ready to illustrate how the second part of the definition of the limit is used to prove that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. Remember, you will control the *y* values by selecting various values for ε, and then ***you will have to find*** the value of δ that gives the corresponding *x* values.

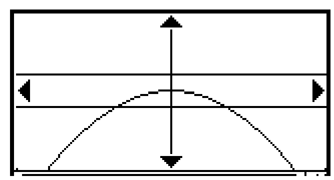
When controlling the *y* values you will be restricting *y* so that it lies between $L - \varepsilon$ and $L + \varepsilon$ where, in this case, $L = 1$. To do this, tap the upper window and define **y2** and **y3** as shown at the right. (The letter e, which is used to denote ε, is entered using the VAR tab on the Keyboard.)



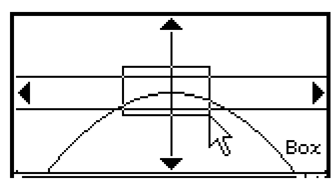
Press M and store a value in e and determine the corresponding *x* values. Let's start by letting $\varepsilon = 0.02$. To do this, store 0.02 in the variable e. Then return to the Graph screen by tapping mW.



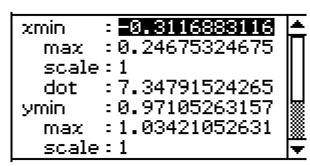
When you graph the functions, the *y* values of $f(x) = \frac{\sin x}{x}$ will be restricted by the horizontal lines. That is, $|f(x) - L| < \varepsilon$ where, in this case, $\varepsilon = 0.02$ and L = 1.



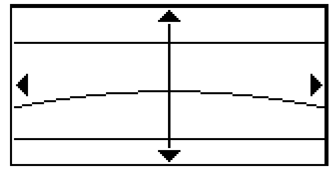
Your task is to find a value for δ such that $a - \delta < x < a + \delta$ where, in this case, $a = 0$. This can be done using **Zoom Box** to create the box so that the horizontal lines are contained in the box and the graph spans from the left side of the box to the right side. This gives $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.



After graphing the function in the box, tap 6 to display the graphing window so you can see what value should be taken for δ. Since in our example $a = 0$, we can take $\delta = 0.2$ in order to get $0 < |x - a| < \delta$. (Why?)



To check this out, set **xmin = -0.2** and **xmax = 0.2** and then graph the function. We see that we have indeed satisfied the condition that the function is between the horizontal lines (i.e.,$|f(x)-L|<\varepsilon$) whenever $-0.2<x<0.2$ (i.e., $0<|x-a|<\delta$).



Notice that in the definition of the limit, the value for δ is not unique. All you had to do was find one such value. If you constructed your box differently than in this activity, as you most likely did, then your value for δ will be different. This is perfectly OK provided that your box is contained within the horizontal lines and the graph spans from the left edge of the box to the right edge.

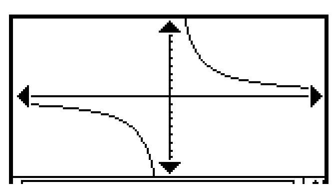
In terms of the ClassPad graphics calculator, we can rephrase the definition of a limit as follows:

> $\lim_{x\to a} f(x)=L$ if for any two horizontal lines $y=L-\varepsilon$ and $y=L+\varepsilon$ we can construct a **Zoom Box** containing these lines and having the property that the graph of $y=f(x)$ continuously spans from the left to the right of the box, with a possible a hole in the graph at $x=a$.

If for some value of ε it is impossible to find such a **Zoom Box**, then the limit does not exist. This is illustrated in the following example.

EXAMPLE 3. $\lim_{x\to 0}\frac{1}{x}$ does not exist.

The graph of $f(x)=\frac{1}{x}$ appears at the right for $-1\le x\le 1$ and $-10\le y\le 10$. We can see from the graph that there is no way we can draw a box containing $x=0$ and having the property that the graph continuously spans from the left to the right of the box, with a possible a hole in the graph at $x=0$. So the limit does not exist.



EXERCISES:

1. Find the largest value for δ, accurate to three decimal places, such that $\left|\frac{\sin x}{x} - 1\right| < 0.01$ whenever $0 < |x - 0| < \delta$.

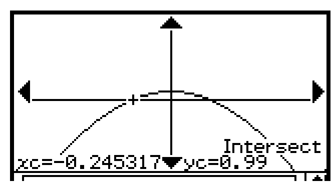
2. Given $\varepsilon = 0.01$, find a value for δ in the definition of the limit showing that $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$.

3. Show that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

SOLUTIONS:

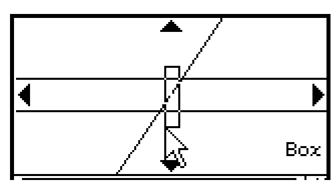
1. **Answer:** $\delta = 0.245$, accurate to three decimal places.

   Store 0.01 in e. Set the Window to $-1 \le x \le 1$ and $0.9 \le y \le 1.1$, or similar settings, and graph the functions. Because of the symmetry of the graph, you can use **Analysis G-Solve Intersect** to find the intersection of the function and the line $y = 1 - .01$.

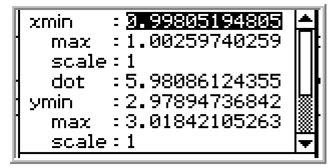


2. **Answer:** Any value of $\delta \le 0.00333$, accurate to five decimal places.

   Set **y1** $= \frac{x^3 - 1}{x - 1}$, **y2** = 3 - e, and **y3** = 3 + e. The value e = .01 should already be stored in the calculator from the previous exercise. Since $x \to 1$, set **xmin = .95** and **xmax = 1.05**. And since you want to show that the limit is 3, set **ymin = 2.95** and **ymax = 3.05**. Graph the functions and construct an appropriate **Zoom Box**.



Then tap 6 to display the graphing window so you can see what value should be taken for δ. Recall that in this problem $x \to 1$. Since **xmin** is approximately 0.01 units away from 1 and **xmax** is only about 0.002 units from 1, we should take $\delta = 0.002$.



3. The graph of $f(x) = \dfrac{|x|}{x}$ appears at the right for $-1 \le x \le 1$ and $-2 \le y \le 2$. ("absolute value" is in the 9 menu on the Keyboard.) We can see from the graph that there is no way we can draw a box containing $x = 0$ and having the property that the graph continuously spans from the left to the right of the box, with a possible a hole in the graph at $x = 0$. So the limit does not exist.