

Finding the Fundamental Matrix for an Absorbing Markov Chain Ben Klein

CALCULATORS: *Casio ClassPad 300*

Student Exercises

The transition matrix below represents a Markov Chain in which States I, II and III are absorbing.

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 1/4 & 1/8 \\ 0 & 1 & 0 & 1/2 & 1/4 \\ 0 & 0 & 1 & 1/8 & 1/8 \\ 0 & 0 & 0 & 1/8 & 1/8 \\ 0 & 0 & 0 & 0 & 3/8 \end{bmatrix}$$

Recall that we denote the 2 by 2 matrix in the lower right hand corner of \mathbf{P} as \mathbf{R} and that we denote the 3 by 2 matrix that lies above \mathbf{R} as \mathbf{S} . i.e. \mathbf{R} contains the transition probabilities from the nonabsorbing states to the nonabsorbing states and \mathbf{S} contains the transition probabilities from the nonabsorbing states to the absorbing states. With this notation, we can write \mathbf{P} as follows, where \mathbf{I} (in this case) denotes a 3 by 3 identity matrix and $\mathbf{0}$ (in this case) denotes a 2 by 3 matrix of 0's.

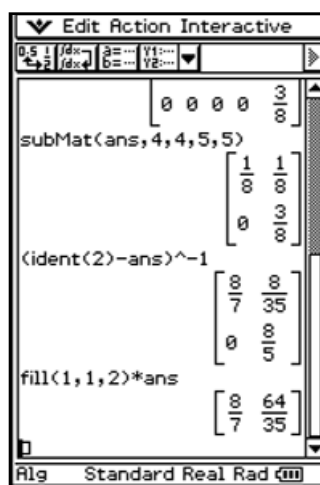
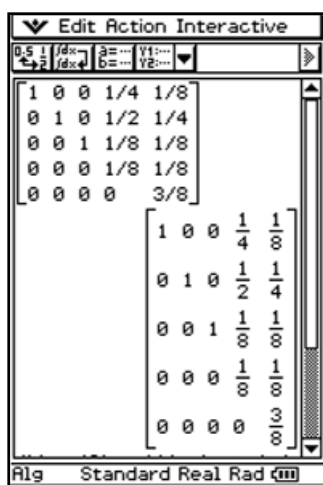
$$\mathbf{P} = \begin{bmatrix} \mathbf{I} & \mathbf{S} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}$$

Recall that there are two important matrices associated with the transition matrix. First, we have \mathbf{F} , the fundamental matrix. \mathbf{F} is the inverse of $\mathbf{I} - \mathbf{R}$, where this \mathbf{I} is the identity matrix with the same dimensions as \mathbf{R} . The column sums of \mathbf{F} give the expected number of transitions prior to absorption for each nonabsorbing state.

Then, we have the stable matrix, $\mathbf{SF} = \mathbf{S}(\mathbf{I} - \mathbf{R})^{-1}$. The entry in, for example, row 3 and column 2 of \mathbf{SF} gives the probability that if the system starts in the second nonabsorbing state (i.e. State V), the system will be absorbed in the third absorbing state (i.e. State III).

You can use your *ClassPad* to do all of the calculations described above.

In the screen capture on the left, we see the original transition matrix. In the screen capture on the right, we see the calculation of the fundamental matrix and the sums of its columns.



Finding the Fundamental Matrix for an Absorbing Markov Chain *continued*

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Student Exercises

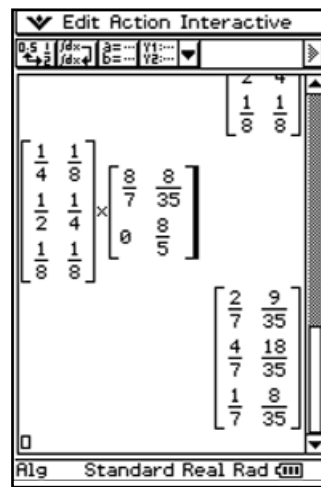
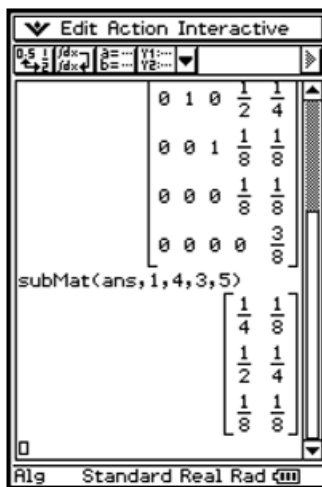
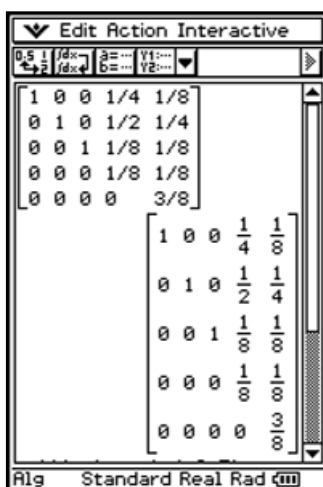
The syntax for the functions **subMat** and **fill** that you see in the screen captures above is:

subMat(matrix, beginning row, beginning column, ending row, ending column)

so that if "ans" is **P**, subMat(ans,4,4,5,5) is **R**

fill(value, number of rows, number of columns)

Next, the screen capture on the left shows the original transition matrix, the screen capture in the middle shows how to create **S**, and the screen capture at the right shows the computation of the stable matrix. You will note that the sums of the columns in the stable matrix are 1, as they must be.



Thus, we see that if our Markov Chain starts in the second nonabsorbing state, i.e. State V, on average $64/35$ transitions will occur prior to absorption and the system will be absorbed in State III with probability $8/35$.

EXERCISE

Suppose the Markov Chain with transition matrix starts in State IV.

$$\begin{bmatrix} 1 & 0 & 1/3 & 1/4 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1/2 & 1/4 \\ 0 & 0 & 1/6 & 1/4 \end{bmatrix}$$

What is the probability that the system is absorbed in State I and what is the expected number of transitions that will occur prior to absorption?

Finding the Fundamental Matrix for an Absorbing Markov Chain *continued*

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Answers

From the middle screen capture, we see that the expected number of transitions prior to absorption is $9/4$. From the screen capture at the right, we see that the probability that absorption occurs in State I is $5/8$.

