Understanding and Using Newton's Method

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Newton's method is an algorithm that uses the derivative of a function to approximate the zeros of the function. To understand how and why Newton's method works, let's first concentrate on the derivative of a function. One of the first things you learned about the derivative was that it could be used to find the slope, and thus the equation, of the tangent to a curve at a specified point. And one of the really nice things about the tangent to a curve is that it can, under certain circumstances, be used to approximate the curve.

To see this, consider Figure 1 which displays a curve and its tangent at the point where $x = a$. When you zoom in on the point of tangency, as displayed in Figure 2, you see that it becomes difficult to distinguish between the graph of the curve and the graph of the tangent. And if you zoom in again, as in Figure 3, it becomes even more difficult to distinguish between the curve and the tangent. So in the "window" depicted in Figure 3, the points on the tangent can be used to approximate the points on the curve.

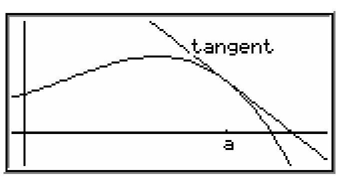


Figure 1

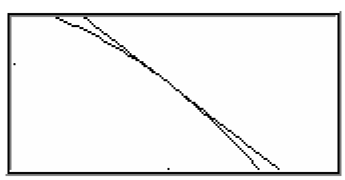


Figure 2

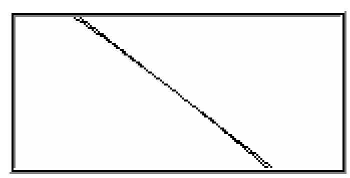
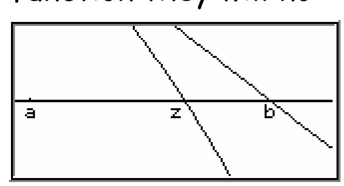


Figure 3

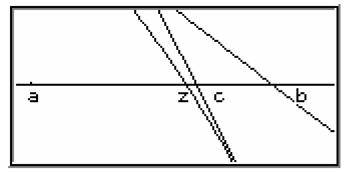
Can you use the points on the tangent to approximate the points on the curve which are not close to the point of tangency? In particular, can you use the zero of the tangent to approximate the zero of a function?

In Figure 1, the zero z of the function and the zero b of the tangent look kind of close, but when you zoom in on the zero of the function they will no longer look very close. This is pictured at the right where a is the x-coordinate of the point of tangency, z is the zero of the function, and b is the zero of the tangent. In this situation, the zero of



the tangent, b, doesn't give a very good approximation of the zero, z, of the function.

But what if you move the point of tangency closer to the zero of the function? Since b is closer to z than a, let's create a tangent to the function at the point where $x = b$. The result is pictured at the right where c is the zero of the new tangent.



As you can see, c is a much better approximation of z than b. And you could probably get an ever better approximation if you continued this process by looking at the zero of the tangent at the point where $x = c$.

This is the basis of Newton's method. That is, to approximate a zero of a function $f(x)$:

1. Pick a value a_0 that is close to the zero of the function.
2. Find the zero a_1 of the tangent to $f(x)$ at the point $(a_0, f(a_0))$.
3. Find the zero a_2 of the tangent to $f(x)$ at the point $(a_1, f(a_1))$.
4. Continue this iterative process.

Each new a_{n+1} should be a better approximation of the zero of the function than the previous approximation a_n.

EXERCISES:

1. Find a formula for a_{n+1}.

2. Use Newton's method and the Sequence application on your calculator to approximate, accurate to two decimal places, the zeros of $f(x) = 2x^3 + x^2 - x + 1$.

3. Use Newton's method to approximate $\sqrt{3}$, accurate to three decimal places.

Answers and Explanations

Using the ClassPad 300

1. Since $f'(a_n)$ is the slope of the tangent to the graph of $f(x)$ at the point $(a_n, f(a_n))$, the equation of this tangent is $y - f(a_n) = f'(a_n)\cdot(x - a_n)$. This tangent crosses the *x*-axis when *y* = *0*. Setting *y* = *0* in the equation of the tangent and solving for *x* gives $x = a_n - \dfrac{f(a_n)}{f'(a_n)}$. In Newton's method, this zero is denoted by a_{n+1}. So Newton's method consists of the iterative function

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}.$$

2. The following steps show how to solve this problem using a ClassPad 300.

   Tap m on the Icon panel at the bottom of the screen and then tap the eActivity application A, as illustrated in the first picture in Figure 1. If the screen is not blank, as in the second picture in Figure 1, tap **File** on the Menu bar and then tap **New** in the drop-down menu. When requested to do so, tap **OK** to clear the screen, as illustrated in the third picture in Figure 1.

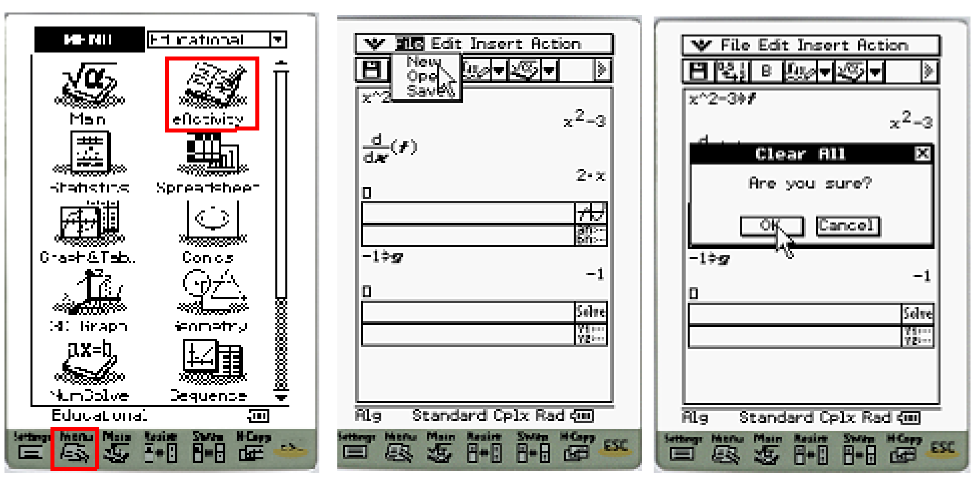
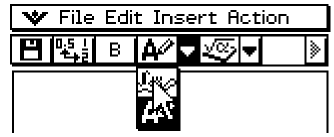


Figure 1

Use the **define** command to define the function $f(x) = 2x^3 + x^2 - x + 1$:

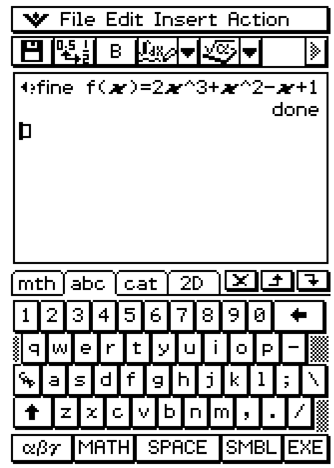
To do this, first tap in the first drop down menu on the Toolbar to change from text entries to calculation entries.



Next press k on the Keypad then tap the 0 tab in the top row of the keyboard. Use the stylus and the keyboard to enter the command **define** and then tap the space bar SPACE at the bottom of the Keyboard.

Then use the Keyboard to enter the letter **f** and then use the Keypad to enter the rest of the formula, as displayed at the right.

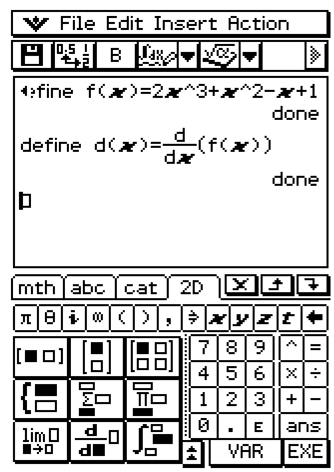
When you are finished entering the formula, press E to store the definition of the function in the calculator.



Store the derivative of f(x) in d(x):

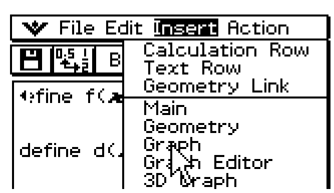
To do this first enter the **define** command, followed by a space. (Note: You can copy this command from the definition of f(x) and then paste it on the screen, or you can enter it from the Keyboard.) Then enter the letter **d** and use the Keypad to enter **(*x*) =**.

Next tap the) tab in the top row of the Keyboard and then tap K at the bottom of the Keyboard. Tap in the last row of the 2D menu and use the Keypad to enter ***x***. Then tap to the left of the box where you enter the function and enter **f(*x*)**. When you are finished, press E to store the definition of the function in the calculator. (Note: Tap 0 to enter **f** from the Keyboard and use the Keypad to enter **(*x*) =**.)

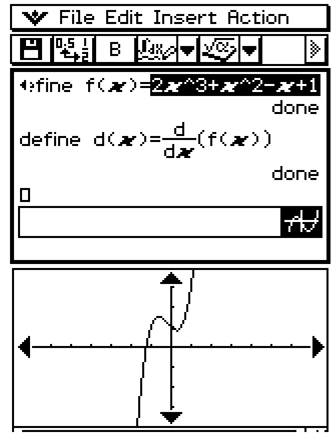


Graph f(x) and estimate where the graph crosses the *x*-axis:

To do this, tap **Insert** on the Menu bar and then tap **Graph** in the drop down menu. A graph window is displayed at the bottom of the screen.

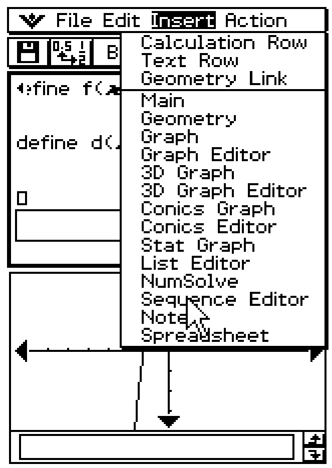


Then highlight and drag the definition of f(*x*) to the graph window to display its graph. From the graph you see that a reasonable approximation of the zero of the function is –1. This value will be used as your first entry, $\mathbf{a_0}$, in the sequence for Newton's method.

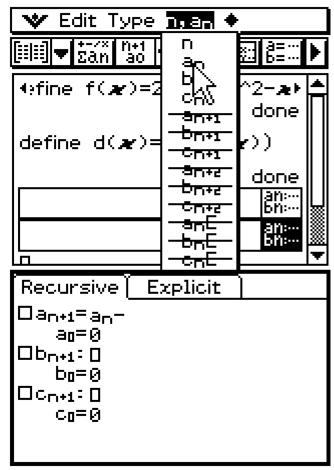


Define the sequence used in Newton's method:

To do this, tap **Insert** on the Menu bar and then tap **Sequence Editor** in the drop down menu.

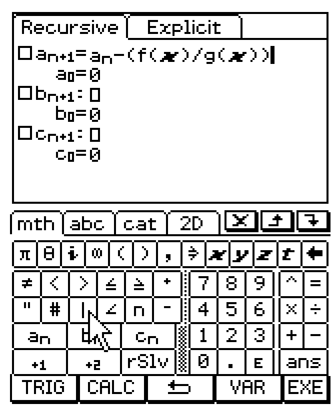


Tap to the left of the box provided for the definition of $\mathbf{a_{n+1}}$ and then tap $\mathbf{n,a_n}$ on the Menu bar. Tap $\mathbf{a_n}$ in the drop down menu to start the definition of Newton's method and then press on the Keypad.

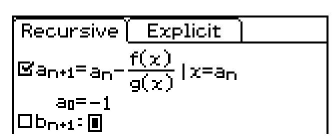


Press k and use the Keyboard and Keypad to enter **(f(*x*)/d(*x*))**. (Note: only the letters need to be entered using the Keyboard, all other symbols can be enterer from the Keypad.)

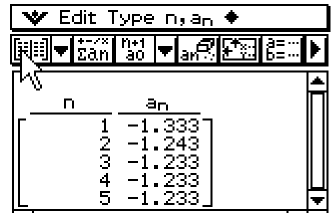
To change ***x*** to $\mathbf{a_n}$, as required by Newton's method, press 9 on the top line of the Keyboard and then press OPTN on the last line of the Keyboard. Then press U to insert the "with" command. Then enter ***x* =**, and then use the menu at the bottom of the screen to enter the symbol $\mathbf{a_n}$. When finished, press E.



Use the Keypad to enter –1 as the initial value of the sequence, and then press E.



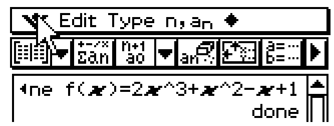
Display the results of using Newton's method:



To do this, tap 2 on the Toolbar to display the results of using Newton's method to find the zero of the function.

Conclusion: From the table it is apparent that the zero, accurate to two decimal places, is -1.23.

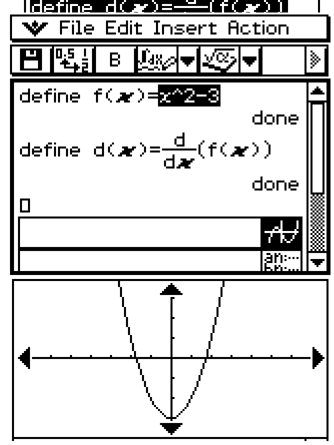
3. Since $\sqrt{3}$ is the root of the function $f(x) = x^2 - 3$, you can simply edit the definitions of **f(x)** and **d(x)** stored in your calculator and use the formula for Newton's method that is already stored in sequence $\mathbf{a_{n+1}}$. Here's how you do this.



Close the display in to top window by pressing 0on the Menu bar and then pressing **Close** in the drop down menu.

Highlight the definition of **f(x)** and use the Keypad to enter the new definition $x^2 - 3$. When finished, press E.

Highlight the definition of the derivative and then press E to calculate the derivative of the newly entered function.



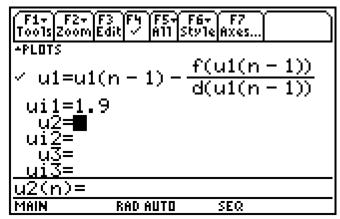
Look at the graph of the redefined function to determine where it crosses the *x*-axis by repeating the following steps you used with the original function:

Then highlight and drag the definition of f(*x*) to the graph window to display its graph. From the graph you see that a reasonable approximation of the zero of the function is -1. This value will be used as your first entry, $\mathbf{a_0}$, in the sequence for Newton's method.

To view the graph of the redefined **f(x)**, return the calculator to Function mode by pressing 3B♦÷ and then press ∞□ to display the graph.

From the graph you see that a possible value for $\mathbf{x_1}$ in Newton's method is 2. A better guess may be 1.9. To enter this guess in the definition of Newton's method you must change the value of **ui1** in the Sequence editor.

To do this, press 3Bψ÷ to put the calculator back in Sequence mode and then press ∞□ to enter the Sequence editor. Then press ∆÷ to place the previous value of **ui1** on the command line, enter the new value, and then press ÷.



To view the values of the sequence, starting with the 100th term, press ∞□. It will take awhile for the table to display. From the table you see that the square root of 3 is, accurate to three decimal places, 1.732.