

Finding the Stable Distribution for a Regular Markov Chain Ben Klein

CALCULATORS: *Casio ClassPad 300*

Student Exercises

If \mathbf{P} is the transition matrix for a regular Markov Chain, then if \mathbf{P} is raised to a large power, the resulting matrix will have constant columns and the probabilities in a given column represent the stable distribution of the Markov Chain.

EXAMPLE: If $\mathbf{P} = \begin{bmatrix} 1/5 & 2/5 \\ 4/5 & 3/5 \end{bmatrix}$, then the screen capture below shows that $\mathbf{P}^{50} = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$

so that the stable distribution of the given Markov Chain is 1/3 for State I and 2/3 for State II. Note that we did the computation in decimal format; the fractions would be huge and not easy to interpret.



It is not always convenient to find the stable distribution of a Markov Chain by raising its transition matrix to a large power. Luckily, we have an algebraic equivalent. If \mathbf{P} is the transition matrix of a regular Markov Chain and \mathbf{X} (a column vector) is its stable distribution, then $\mathbf{P}\mathbf{X} = \mathbf{X}$ and the sum of the entries in \mathbf{X} is 1.

Thus, to determine \mathbf{X} we can solve $(\mathbf{P} - \mathbf{I})\mathbf{X} = \mathbf{0}$ where \mathbf{I} is the identity matrix with the same shape as \mathbf{P} and $\mathbf{0}$ denotes the column vector of 0's, remembering that the sum of the entries in \mathbf{X} must be 1. Thus, in the example above, we have:

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ with } x_1 + x_2 = 1 \text{ and } \begin{bmatrix} -4/5 & 2/5 \\ 4/5 & -2/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This system of equations translates into a system of three linear equations in two unknowns which can be represented by the following augmented matrix which can then be put into reduced form which then gives the stable distribution for the Markov Chain.

$$\text{Augmented Matrix: } \begin{bmatrix} -4/5 & 2/5 & 0 \\ 4/5 & -2/5 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{Reduced Form: } \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{bmatrix}$$

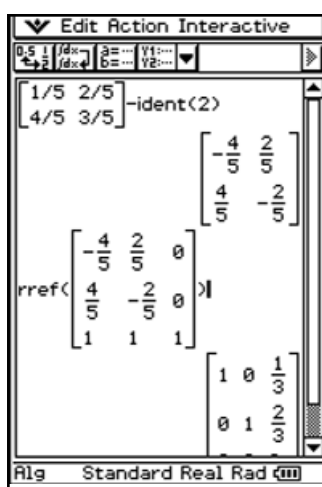
Finding the Stable Distribution for a Regular Markov Chain *continued*

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Student Exercises

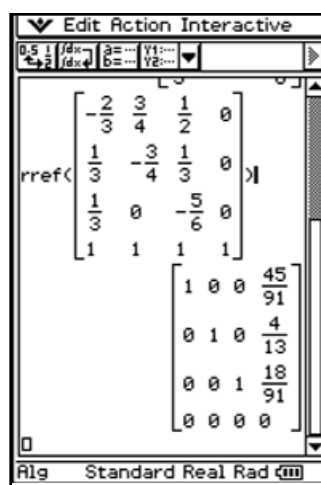
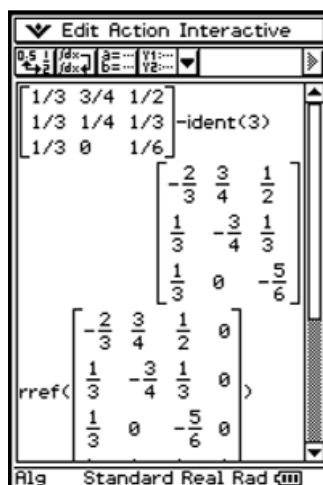
Setting up the augmented matrix and obtaining the stable distribution is easy with your *ClassPad*. First you enter the transition matrix **P**. [Best done using the 2D menu] Then: (1) find the matrix **P – I** [using the ident function in Matrix-Create]; (2) add a row to the bottom and a column on the right of this matrix [using the appropriate functions in the 2D menu]; (3) fill 1's in the bottom row and 0's in the rightmost column (except in the bottom row); (4) apply rref and read off the stable distribution.

The following screen capture shows the results of these operations. The bottom row of the bottom matrix in the screen capture is a row of 0's.



Here another example, with the larger transition matrix $\begin{bmatrix} 1/3 & 3/4 & 1/2 \\ 1/3 & 1/4 & 1/3 \\ 1/3 & 0 & 1/6 \end{bmatrix}$.

The screen captures below show how to find the stable distribution for this matrix and that the stable probabilities are 45/91, 4/13 and 18/91 for States I, II and III respectively.



Name _____ Date _____

Finding the Stable Distribution for a Regular Markov Chain *continued*

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Student Exercises

EXERCISES:

Find the stable distributions for Markov Chains whose transition matrices are given below.

$$\begin{bmatrix} 0 & 1/3 \\ 1 & 2/3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1/5 & 1/4 \\ 1/2 & 2/5 & 1/2 \\ 1/2 & 2/5 & 1/4 \end{bmatrix}$$

$$\begin{bmatrix} 1/3 & 0 & 1/4 & 0 \\ 0 & 1/4 & 1/4 & 1/2 \\ 1/3 & 1/2 & 1/4 & 0 \\ 1/3 & 1/4 & 1/4 & 1/2 \end{bmatrix}$$

Finding the Stable Distribution for a Regular Markov Chain *continued*

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Answers

See the screen captures below.

