Investigating the Expected Value of a Discrete Random Variable

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CALCULATORS: Casio: *fx-9750G Plus* • Casio: *CFX-9850G Series*

Teaching Notes/Lesson Plan

Setting the background

In a certain state, a player can play Keno by selecting from one to ten numbers out of the set 1, 2, 3, …, 80. The payout chart below lists the prizes per dollar wagered that can be won depending on the number of "spots" (numbers) that match the ten numbers randomly generated by the game.

1-spot game: Match 1; prize $2

2-spot game: Match 2; prize $10

3-spot game: Match 3; prize $25
Match 2; prize $2

4-spot game: Match 4; prize $50
Match 3; prize $5
Match 2; prize $1

5-spot game: Match 5; prize $300
Match 4; prize $15
Match 3; prize $2

6-spot game: Match 6; prize $1000
Match 5; prize $50
Match 4; prize $5
Match 3; prize $1

7-spot game: Match 7; prize $2500
Match 6; prize $100
Match 5; prize $15
Match 4; prize $3
Match 3; prize $1

8-spot game: Match 8; prize $10,000
Match 7; prize $500
Match 6; prize $50
Match 5; prize $10
Match 4; prize $2

9-spot game: Match 9; prize $25,000
Match 8; prize $2500
Match 7; prize $100
Match 6; prize $20
Match 5; prize $5
Match 0; prize $2

10-spot game: Match 10; prize $100,000
Match 9; prize $4000
Match 8; prize $400
Match 7; prize $50
Match 6; prize $10
Match 5; prize $2
Match 0; prize $4

In the long run, all options will cause the bettor to lose money. Which option offers the opportunity to lose the least amount of money?

The probability of selecting "s" spots (numbers) and matching exactly "m" of the spots when "r" numbers are randomly generated ($r \geq s$) from a set of "N" numbers id given by the following formula.

$$\mathbf{p(s, m)} = \frac{(s\,\mathbf{C}\,m) \times [(N - s)\,\mathbf{C}\,(r - m)]}{N\,\mathbf{C}\,r}$$

In this state, 10 numbers (*r*) are randomly selected out of the set of 80 numbers (*N*).

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The mean, or expected value, of a discrete random variable x is the sum of the products of the outcomes and their associated probability.

$$\mu = \mathbf{E}(x) = \sum\{x \times [\mathbf{p}(x)]\}$$

In the one-spot game, you win $2 if the number that you select matches any of the 10 numbers randomly generated. If your spot does not match any of the numbers randomly generated, you win nothing. These are the two outcomes that are possible. The probability of the first is calculated by using the formula presented earlier. The probability of the second is 1 – (probability of first).

$$\mathbf{p}(1, 1) = \frac{(1\ C\ 1) \times [(79)\ C\ (9)]}{80\ C\ 10}$$

The following keystrokes make the combinations function available.

MENU; **RUN(1)**; **OPTN**; **F6**; **PROB** (**F3**)

The following keystrokes produce the probability.

1; **nCr** (**F3**); **1**; × **79**; **nCr** (**F3**); **9**; ÷ ; **80**; **nCr** (**F3**); **10**; **EXE**

This produces the probability of winning $2: 0.125. When this game is played many times, the average expected outcome is: ($2 × 0.125 + $0 × 0.875) = $0.25. This means that for every dollar wagered the bettor expects to win twenty-five cents. The difference between the return and the amount wagered is -$0.75. Over the long run, a bettor should expect to lose $0.75 per game for each game played.

The expected outcomes for the 2- and 3-spot games can be calculated in a similar manner. Those outcomes are losses of $0.8576 and $0.8861, respectively. Beyond that point there is a simpler way to do the repeated calculations: Lists!

Select an unused drawer in the file cabinet: **MENU**; **LIST(4)**; **OPTN**; **SHIFT SET UP** . Select a drawer (File 1 (**F1**) through 6 (**F6**); **EXE**) until you find one that does not contain lists of data.

Enter the number of matches that must be made in List 1. The 4-spot game is used for demonstration purposes, and 4, 3, and 2 are entered into List 1. Cursor to the title of List 2. The title will be highlighted. Access the combinations function as before. The same probability formula is entered, but certain values vary and are tied in to the values found in List 1.

Teaching Notes/Lesson Plan

Enter the following. spots in game (**4**, for the 4-spot game); **nCr** (**F3**); **EXIT**; **LIST** (**F1**); **List** (**F1**); **1** (where the matches are located); ×; **EXIT**; **PROB**; (80 – spots), (which is 76 for the 4-spot game); **nCr** (**F3**); (; **10**; **–**; **EXIT**; **LIST** (**F1**); **List** (**F1**); **1**;); ÷; **80**; **EXIT**; **PROB** (**F3**); **nCr** (**F3**); **10**; **EXE**.

Enter the respective prizes in List 3.

The products of the prizes and their probabilities will now be calculated. Cursor to the title of List 4. Enter the following: **OPTN**; **LIST** (**F1**); **List** (**F1**); **2**; ×; **List** (**F1**); **3**; **EXE**.

The expected value is the sum of List 4, because the "prize" for any other outcome is $0. Regardless of the probability of the outcome(s), the product will be zero, which will not increase the sum of List 4.

Enter the following: **MENU**; **RUN** (**1**); **OPTN**; **LIST** (**F1**); **F6** (▶); **F6** (▶) ; **Sum**; **F6** (▶) ; **List**; **4; EXE**.

The expected value is approximately $0.10. Over the long run, a bettor should expect to lose $0.90 per game for each game played.

It appears that the amount lost per game played increases as the number of spots played increases. Is this trend constant? The above process can be used to calculate the expected outcomes for the other spot options. For the 5- through 8-spot games the respective losses per dollar waged are as follows: $0.964; $0.966; $0.951, and $0.981.

Except for the minor hiccup in the 7-spot game, the expectation becomes more and more bleak. However, something surprising happens beyond this point. For the 9-spot game, the bettor should only expect to lose $0.433 per dollar waged. As surprising as that may seem, the result for the 10-spot game is more shocking: a loss of only $0.031! This turnabout is heavily influenced by the same particular option of winning in each of the games: matching 0.

Perhaps the moral of this story is: If you insist on throwing away money by gambling, try throwing away as little as possible: play the 10-spot game. Although the probability of winning $100,000 (about 6×10^{-13}) is relatively far less that the probability of being struck by lightning, the potential reward is more thrilling than trying to win $10,000 in the 8-spot game (the worst of all choices), and you will be throwing away fewer dollars in your futile attempt to come out a winner.