CLEMSON MIDDLE SCHOOL MATHEMATICS PROJECT
UNIT 1: DEVELOPING NUMBER SENSE

PROBLEM 1: HOW FAST DOES IT GROW?

Take a normal sheet of paper eight-and-a-half inches wide and eleven inches long. Tear it in half, putting the two halves on top of each other. Tear this in half, again putting the two halves on top of each other. Continue this process.

A. If you would do this 50 times, estimate the height of the stack of papers.
B. Estimate the area of one of the pieces of paper in the stack.
C. Calculate the height of the stack of papers.
D. Calculate the area of one of the pieces of paper in the stack.

MATERIALS

Casio *Algebra FX 2.0* Graphing Calculator

EXTENSION

Suppose you invest your money so that it doubles every 8 years. Which would be worth more when you turn 65: investing $1,000 when you are 9-years-old or investing $10,000 when you are 41-years-old? Support your answer.

REFERENCE: Thanks to Dr. Joe Garofalo at the University of Virginia and the Impact Project group.

ONE SOLUTION TO PROBLEM 1: HOW FAST DOES IT GROW?

A. If you would do this 50 times, estimate the height of the stack of papers.

Answers will vary significantly. Insist that students write down their estimates. The teacher should then record the answers for the class to see the range of answers.

B. Estimate the area of one of the pieces of paper in the stack.

Again, answers will vary. As before, have students write down their estimates and then display them for everyone to see.

C. Calculate the height of the stack of papers.

We will assume that we can compress 250 sheets of paper into a stack one-inch high. First, we will determine the number of sheets in the stack. To see how tall this is, we will then convert it to familiar units.

To determine the number of sheets in a stack, we will begin a table and look for a pattern.

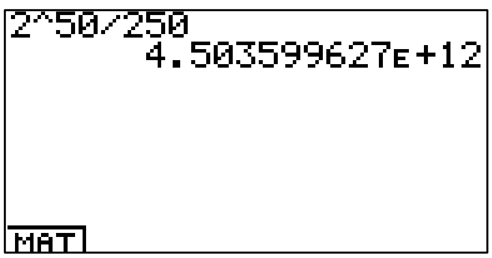
NUMBER OF TEARS	NUMBER OF SHEETS
0	1
1	2
2	2 ? 2 ? 2^2 ? 4
3	4 ? 2 ? 2^3 ? 8
4	8 ? 2 ? 2^4 ? 16
5	16 ? 2 ? 2^5 ? 32
6	32 ? 2 ? 2^6 ? 64
7	64 ? 2 ? 2^7 ? 128

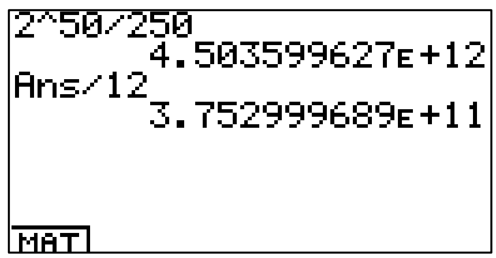
If we look at the exponential form of the numbers in the table, we can determine that after n tears, we should have 2^n sheets of paper. Thus, after 50 tears, we will have 2^{50} sheets of paper.

How big is 2^{50}, and how tall would a stack of papers with 2^{50} sheets of paper in it be? We will use the calculator to help us determine the answer.

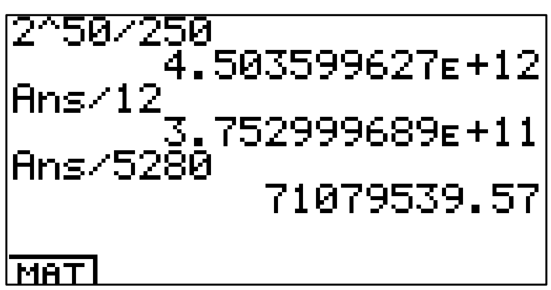
From the MAIN MENU, choose RUN-MAT mode.

- We know 2^{50} represents the number of sheets of paper in the stack. We have assumed that 250 sheets make a stack one-inch high. Consequently we want to divide 2^{50} by 250 to determine how tall the stack is in inches. Type in 2, [^] for the exponent, 50, ?, 250, and press [EXE]. See below left. The result indicates that the stack is approximately $4.5036 ? 10^{12}$ inches high.
- If we divide by 12, our result will be expressed in feet. Using the calculator's automatic feature, simply press ?, 12, and [EXE]. See below right. We find that the result, approximately $3.75 ? 10^{11}$ feet, is still too big to really understand.





- If we now divide by 5,280, our answer will be expressed in miles. Again using the automatic feature, simply press ?, 5280 and [EXE]. See below. This time our result, approximately 71,000,000 is something easier to grasp.



The distance from the earth to the moon is approximately 240,000 miles. Our stack of papers far exceeds that! In fact, the stack of papers reaches more than two-thirds of the way from the earth to the sun. Not many people expect this result. Through experience, most people, not only students, learn to think linearly. This problem, however, calls upon us to think exponentially, something few of us are able to do intuitively.

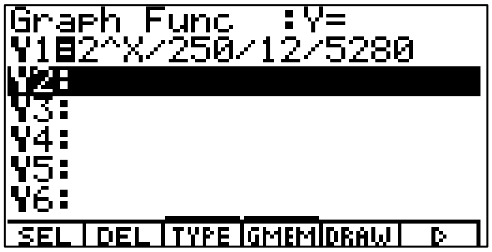
To help conceptualize what is happening with the magnitude of this number, we will now explore the problem graphically. First we need to identify our variables.

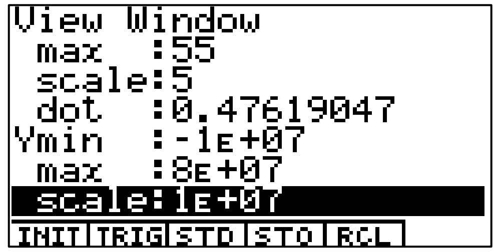
Let x represent the number of tears we have made. Note that x will be a whole number from 0 up to 50. Let y represent the height of the pile, measured in miles. To express y in terms of x, we will generalize what we have just done, using our variable x instead of 50. That is, to find the height of the pile in miles (y) after making x tears, $y \; ? \; 2^x \; ? \; 250 \, ? \, 12 \, ? \, 5280$.

From the MAIN MENU, choose GRPH-TBL.

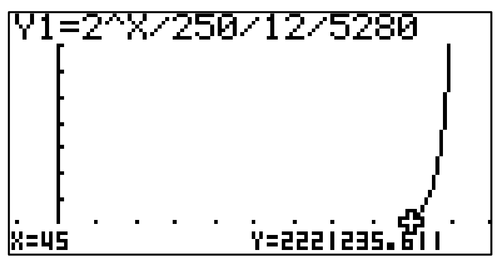
- If there are any functions already entered, either deselect them by highlighting them and pressing [F1] or delete them by highlighting them and pressing [F2] followed by [EXE] for "yes".
- Type in the function, using [X,?,T] r the x variable and [^] for the exponent. Then press [EXE] . See below left.
- Before viewing the graph, we want to set a window so that we can see the graph clearly. Press [SHIFT] [OPTN] to obtain the V-Window (viewing window) screen. The values suggested here are arbitrary. Type in –5 and press [EXE] for the Xmin, 55 and [EXE] for max, and 5 and [EXE] for the scale. Use the down arrow on the disc to bypass the "dot." We'll have to use scientific notation to enter our large numbers. For a minimum y-value of $?\,10{,}000{,}000$, type in –1 [EXP] 7 for Ymin and press [EXE] . For a maximum y-value of 80,000,000, type in 8 [EXP] 7 for max and press [EXE] . To put

tick marks every 10,000,000, type in 1 EXP 7 for scale and press EXE . See below right for a portion of the screen.





- To view the graph, press ESC to return to the main graphing screen followed by F5 to see the graph. See below left.
- You can trace through the graph by pressing F1 and then using the right and left arrow keys on the disc to move around. For example, the point 45, 2.22E6, shown below right, indicates that after 45 tears have been made, the stack is 2,221,235.611 miles high. Note how the graph "hugs" the *x*-axis until many, many tears have been made. In other words, it isn't until the last few tears have been made that the pile reaches into the sky!



D. Calculate the area of one of the pieces of paper in the stack.

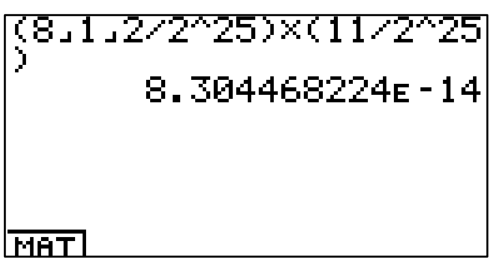
The original dimensions were $8\frac{1}{2}$ by 11 eleven inches. After the first tear, one of these dimensions is divided by 2; assuming we alternate between length and width, after the second tear the other dimension would also be divided by 2. If we continue alternating dimensions in this fashion for the 50 tears, the width of one piece will be $8\frac{1}{2} ? 2^{25}$ inches and the length will be $11 ? 2^{25}$ inches. To find the area of one of the pieces of paper after the last tear, we want to multiply the length and width. We will use the fraction bar to assist us. From the MAIN MENU choose RUN-MAT. Then,

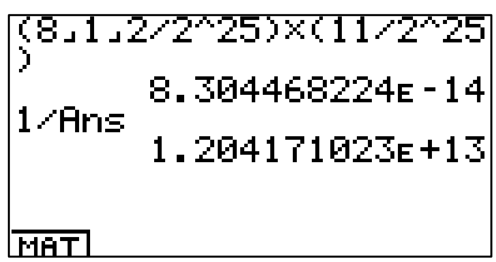
- Press left parenthesis, 8, [a b/c] for the fraction key, 1, [a b/c] again to separate within the fraction, 2, ?, 2, [^], 25, and the right parenthesis. Then press ?, left parenthesis, 11, ?, 2, [^], 25, and the right parenthesis. Finally press [EXE]. See below left.

The area is approximately $8.30 ? 10^{?14}$ square inches. How small is this? If we divide this value into 1, we can determine how many of these squares it would take to fill in one square inch.

- Press 1, ?, [SHIFT], and [(-)] (the last two keys bring up Ans). Press [EXE].

  See below right. This tells us it would take approximately $1.20 ? 10^{13}$ pieces of paper to fill up one square inch.. That's 12,000,000,000,000 (12 trillion) pieces of paper. Needless to say, this theoretical result is not practically possible, and the number is far larger than most people can fathom!





PROBLEM 2: CRICKET ON THE NUMBER LINE

A cricket is on the ground and notices that it is beginning to rain. It wishes to hop to a spot beneath a tree to get out of the rain. You set up a number line, labeling the beginning point 0 and the ending point 1. As the cricket gets wet, it finds that it can only hop half the remaining distance each time.

You are to explore the cricket's journey.

A. Where is the cricket after 1 hop? After 2 hops? After 7 hops?

B. Does the cricket ever make it to the point beneath the tree? Explain.

C. Find a pattern in the fraction of the distance the cricket has gone after each hop. Express the results with exponents.

D. Write a formula for the fraction of the distance the cricket has hopped in terms of the number of hops it has made.

E. When will the cricket pass the point $\frac{511}{512}$? $\frac{5}{6}$? $\frac{9}{10}$?

F. Use a graph to explore the relationship between the number of the hop and the remaining distance and between the number of hop and the total distance traveled.

MATERIALS

Casio *Algebra FX 2.0* Graphing Calculator

EXTENSION

Express the results from above in decimals and percents. Change the problem so that the cricket hops one-third of the remaining distance each time. Explore the results in fractions, decimals, and percents. Write a paragraph discussing your results.

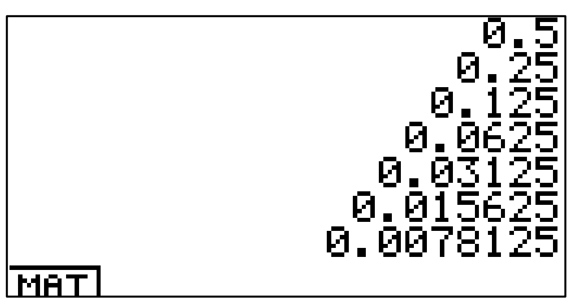
REFERENCE: "Patterns and Functions," Curriculum and Evaluation Standards for School Mathematics Addenda Series, Grades 5-8, NCTM, 1996, p.34.

ONE SOLUTION TO PROBLEM 2: CRICKET ON THE NUMBER LINE

A. Where is the cricket after 1 hop? After 2 hops? After 7 hops?

After one hop, the cricket will be at one-half. After two hops, the cricket will have $\frac{1}{4}$ of its trip left; that is, it will be at $\frac{3}{4}$. After 7 hops, it will be at the point $1 - \frac{1}{2^7}$, with $\frac{1}{2^7}$, or $\frac{1}{128}$, representing the distance it has left. To show the distance the cricket has left on a calculator, choose RUN-MAT from the MAIN MENU. Then,

- Press [CTRL] [F3] for the SET UP. Use the down arrow on the disc until Display is highlighted. Press [F3] until Nrm2 shows. (Under Nrm1, the calculator will display in scientific notation with numbers smaller than .01; with Nrm2, scientific notation won't appear until you reach numbers smaller than 10^{-9}. For large numbers, there is no difference between the settings.) Press [EXE] to return to the home screen and press [AC/on] if necessary to clear home screen.
- Press 1 and [EXE] .
- Press the division symbol, 2, and [EXE] . This represents the distance left after one hop.
- Press [EXE] six more times until it has been pressed a total of 7 times. This number, 7.8125×10^{-3}, or 0.0078125 units, is the distance the cricket has left to travel. See below.



B. Does the cricket ever make it to the point beneath the tree? Explain.

If one continues to press [EXE], eventually the calculator displays a number extremely close to 0. Theoretically, the number continues to become smaller, but never reaches 0, suggesting that the cricket will never make it to the tree.

C. Find a pattern in the fraction of the distance the cricket has gone after each hop. Express the results with exponents.

The distance the cricket has traveled after each hop is shown in the following sequence: $? \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \frac{127}{128}, ... ?$. In exponential form, this sequence could be written as $? \frac{2^1 ? 1}{2^1}, \frac{2^2 ? 1}{2^2}, \frac{2^3 ? 1}{2^3}, \frac{2^4 ? 1}{2^4}, \frac{2^5 ? 1}{2^5}, \frac{2^6 ? 1}{2^6}, \frac{2^7 ? 1}{2^7}, ... ?$.

D. Write a formula for the fraction of the distance the cricket has hopped in terms of the number of hops it has made.

Using exponents, the distance the cricket has left after *n* hops can be expressed as $\frac{1}{2^n}$. Consequently, the distance the cricket has actually hopped is $1 ? \frac{1}{2^n}$, or $\frac{2^n ? 1}{2^n}$.

E. When will the cricket pass the point $\frac{511}{512}$? $\frac{5}{6}$? $\frac{9}{10}$?

There are several methods to tackle this. The cricket will pass $\frac{511}{512}$ when it has less than $\frac{1}{512}$ units less. Similarly, it will pass the other points when it has less than $\frac{1}{6}$ and $\frac{1}{10}$ left to travel, respectively. Since it will have $\frac{1}{8}$ left to go after three hops, it will have passed $\frac{5}{6}$ by then. Similarly, since it will only have $\frac{1}{16}$ to go after four hops, it will have passed $\frac{9}{10}$ by then.

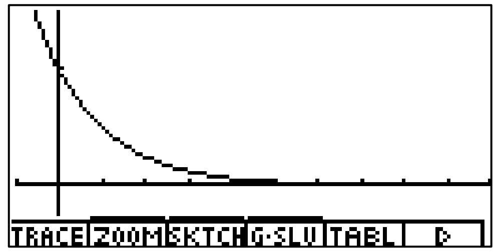
Working with $\frac{1}{512}$ is a little more difficult. If we change it to a decimal, we find that is 0.001953125. Using the technique shown in part A, we find that we reach this point at exactly 9 hops. In other words, it will pass this point on its 10th hop.

F. Use a graph to explore the relationship between the number of the hop and the remaining distance and between the number of hop and the total distance traveled.

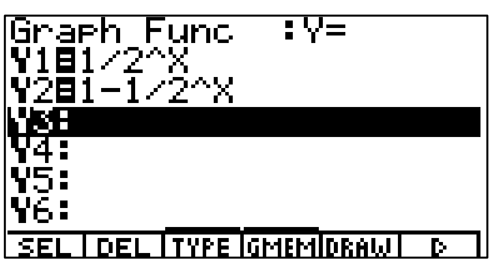
First we'll explore the amount the cricket has left to travel. From the MAIN MENU, call up GRPH-TBL.

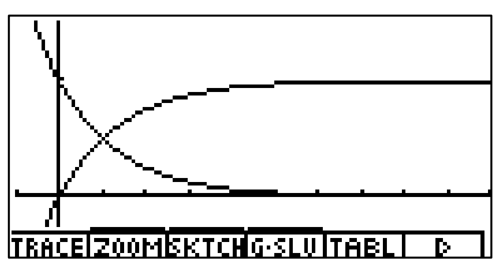
- If any function is there, either delete it (F2 followed by EXE) or deselect it (by pressing F1 , assuming, of course, that it is currently selected). For the screens shown here, all previous functions were deleted.
- We now want to enter the function. Simply type in 1, ? , 2, ^ for the exponent, and the X,?,T key for variable *x*. Then press EXE . See below left.
- Press SHIFT OPTN to set the window. Type in appropriate values for the Xmin, max, and scale, pressing EXE after each. Appropriate values might be –1 (so we can see to the left of the first point), 10, and 1. Then press the down arrow key to set the Ymin, max, and scale. Again, type in appropriate values, pressing EXE after each entry. Appropriate values might be –0.5 (so we can see below the bottom values, 1.5 (so we can see a little above the top of the graph), and 1.

- After the window has been set, press ESC to return to the Graph-Func screen followed by F5 to draw the graph. See below right.



- Press F1 and use the right and left arrow keys on the disc to trace through the graph. Notice that the graph slides downward, settling closer and closer to 0. This tells us that the distance the cricket has left to travel is approaching 0. We'll now also take a look at the distance the cricket has hopped.
- While looking at the graph, press ESC to return to the Graph-Func screen, which is used to enter the functions. (You may need to press ESC a couple of times before you get back to the Graph-Func screen.)
- Use the down arrow key on the disc to move down to Y2. Type in the function that shows the distance the cricket has hopped. We can think of this function as either $\frac{2^x - 1}{2^x}$ or as $1 - \frac{1}{2^x}$. Since the latter may seem simpler, we'll enter this. Type in the values and press EXE (Remember to use the X,?,T button to enter "x".) See below left.
- Press F5 to draw the graph. See below right.

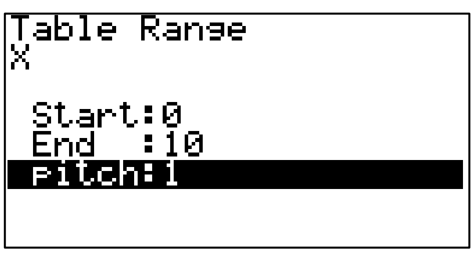


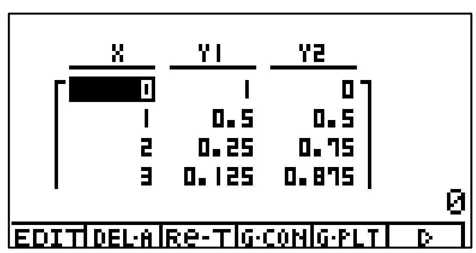


Note that both graphs now appear. The first, the one that slopes downward, shows the amount the cricket has left to hop, a value that gets closer and closer to 0. The second, the graph that slopes upward, shows the amount the cricket has traveled, a value that gets closer and closer to 1. Although it is not important for middle school students to know the term, these graphs have horizontal asymptotes at 0 and 1, respectively.

Although not requested, it might also be informative to explore this with a table. Press ESC to return to GraphFunc screen, press F6 until the menu option RANG appears on the screen. Then press F2 to call up the Table Range screen.

- Since the functions are already be entered in y1 and y2 in the Graph-Func screen, you will set the range for the *x*-values in the table. Appropriate values might be 0 for the start, 10 for the end, and 1 for the pitch. As usual, press EXE after each entry. See below left.
- Press ESC if needed to return to the GraphFunc screen followed by F5 to see the table.
- You can use the right and left arrows on the disc to move back and forth between X, the number of jumps, Y1, the distance remaining, and Y2, the distance jumped. You can also use the up and down arrows to see the values as the number of jumps, represented by X, changes. See below right for the first part of the table.





We have now explored this problem numerically (at the beginning and again at the end with the table), algebraically (with the formulas), and graphically. Looking at a problem in different ways helps students learn at deeper levels and also helps them make connections that they may not otherwise make.

PROBLEM 3: HOW WELL DID YOU DO?

Suppose your math teacher determines your grade by keeping a running total of points. For example, if you score 7 out of 10 on a quiz and 38 out of 40 on a project, your combined grade would be 45 out of 50, or 90%. Now suppose that during the grading period you had earned the following grades:

QUIZZES: 7 out of 12; 8 out of 15; and 7 out of 10

TEST: 97 out of 103

PROJECTS: 35 out of 44 and 33 out of 40

A. Without converting your scores to percents, try to arrange them in order from highest to lowest in terms of how well you did.

B. Now enter the grades in fractional form into a list in your calculator. Sort the list from highest to lowest (descending). Compare your results with part A.

C. How many points did you earn during the grading period? How many points were possible? Change this value to a percent; this represents how well you did over the course of the grading period.

D. If you write all of the scores in fractional form, note that your overall grade is obtained by adding the numerators and adding the denominators. However, you have been taught to add fractions by finding a common denominator and then adding the numerators and leaving the denominator alone. Think about this. Then write a brief paragraph explaining your understanding of what is happening here. (Hint: When you add and subtract fractions in the traditional manner, the meaning of one whole does not change.)

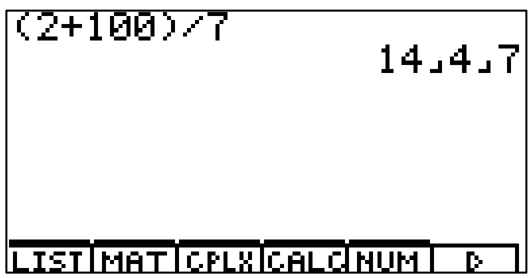
E. Convert each score to a percent. Average the percents. Do you obtain the same average you did in part C? Explain.

PROBLEM 4: MONTHLY REMAINDERS

Remainders can be used to determine the results of things that cycle. For example, consider the days of the week, thinking of Sunday as 0, Monday as 1, Tuesday as 2, Wednesday as 3, Thursday as 4, Friday as 5, and Saturday as 6. If you want to know what day it is 100 days after a Tuesday, simply add 2 (since Tuesday is represented by the number 2) to 100, divide by 7, and check the remainder. The remainder, 4, tells us that it will be the day numbered 4, which is Thursday.

Working this same problem using the *Algebra FX 2.0* Graphing Calculator's fraction key can help develop a deeper understanding in students for the three components of a mixed number. From the Main Menu, choose RUN-MAT. (To clear the screen, press [AC/ON] .)

- Enter the left parentheses, 2, the addition sign, 100, the right parentheses, the divide sign, and 7. Press [EXE] and then the fraction key [a b/c] . See below.



The 14 4 7 on the screen represents the mixed number $14\frac{4}{7}$. Each component of this mixed number has a specific meaning within the context of this problem. The whole number 14 is the number of weeks (groups of 7) in the 102 days. The fraction $\frac{4}{7}$ represents a portion of 1 week. The number 4 gives the day, Thursday, of that week, and the 7 represents the total days found in the week. Encouraging students to make these associations between the numbers and the problem will provide concrete application to the abstract meaning of mixed numbers.

We can work with months the same way, if we divide by 12. This problem is a bit more complicated using the calculator because of the default to lowest term fractions.

However it provides an excellent opportunity to apply the concept of renaming fractions. There is a natural correspondence between the months and numbers. That is, January is 1, February is 2, March is 3, …, and December is 12. Since division by 12 will yield 12 different remainders, namely 0,1,2,…11, the remainder will indicate which month when we divide by 12. Have your students discuss which remainder will indicate December, since 12 is never a remainder. (The remainder 0 implies December.)

We will examine 4 separate problems, each presenting a slightly different scenario for the student to consider.

1) 45 months after March

   Enter (3+45)? 12 into the calculator. Press [EXE] . Since there is no decimal, it is not necessary to press the fraction key. No decimal indicates a remainder 0, because 12 divided evenly into the dividend. Therefore 45 months after March will be December.

2) 130 months after January

   Enter (1+130)? 12 into the calculator. Press [EXE] and since a decimal appears, also press the fraction key [a b/c] . The answer is $10\frac{11}{12}$ meaning there are 10 years (10 groups of 12) in the 131 months and $\frac{11}{12}$ of 1 year, where the 11 indicates November.

3) 150 months after August

   Enter (8+150) ? 12 into the calculator. Press [EXE] and since a decimal appears, also press the fraction key [a b/c] . The answer is $13\frac{1}{6}$. The $\frac{1}{6}$ presents a new concern within this problem. The 13 represents 13 groups of 12 (13 years), but the fraction in its present form does not give us the information we seek. In order to determine which month, the denominator must be 12, because there are 12 months in a year, not 6. So the fraction must be renamed. Encourage discussion among your students about this before you simply tell them how to do it. This problem may give students insight into fraction renaming that actually has a reason. Since the denominator needs to be 12, students will need to perform the following without simplifying the result: $\frac{1}{6} ? \frac{2}{2} ? \frac{2}{12}$. The 2 indicates it will be the month of February.

4) 15 months *before* February

Enter (2-15)? 12 into the calculator. Press [EXE] and since a decimal appears, also press the fraction key [a b/c]. The negative value indicates that the 1 year is *prior* to February rather than *after* it. Even if you have not done positive and negative numbers with your students, you can still discuss this idea with them. Since the years are going *backwards*, the months indicated in the fraction are too. You can adjust for this by subtracting the fractional part from 1. That is, calculate $1 ? \frac{1}{12} ? \frac{11}{12}$. Thus we find that 15 months before February is the month November.

Now try the following problems.

A. What day will it be 40 days after a Wednesday?
B. What month will it be 30 months after August?
C. What time will it be 100 hours after 9:00 am? (Hint: If you use a 24-hour system, you can tell whether it's morning or afternoon. If the remainder is larger than 12, the time is in the afternoon; to determine the hour, simply subtract 12. For example if you obtain a remainder of 18 when you divide by 24, the time is 6:00 pm. A remainder of 12 indicates noon (12:00 pm), and a remainder of 0 indicates midnight (12:00 am). A remainder less than 12 indicates a time in the morning.)
D. What day was it 20 days before Wednesday?
E. What month was it 15 months before February?
F. What time was it 80 hours before 5:00pm?
G. This problem involves a topic called modular arithmetic. Make up at least five good problems of this type. Try them on your friends and family.

TEXT SECTION CORRESPONDENCES

The materials in this module are compatible with the following sections in the listed texts.

TEXT	SECTION
AW – Foundations of Algebra and Geometry (1998)	1.2-3; 2.1; 2.3; 3.3; 4.1; 8.1-2
Glencoe – Mathematics Applications and Connections C1 (1995)	1.7; 2.1; 2.6; 3.10; 6.9-11; 13.5
Glencoe – Mathematics Applications and Connections C2 (1995)	1.8; 2.6; 3.7; 6.4; 14.3-5
Glencoe – Mathematics Applications and Connections C3 (1995)	1.9; 2.6; 6.11; 11.3
Houghton Mifflin – The Mathematics Experience I (1992)	1.2; 1.7; 5.11; 5.13
Houghton Mifflin – The Mathematics Experience II (1992)	1.2-4; 1.6; 1.10; 3.8; 3.10; 4.5; 12.2-3
McDougal Littell – Gateways to Algebra and Geometry (1994)	3.2; 8.2; 11.7; 12.4
Prentice Hall – Middle Grades Mathematics C1 (1995)	1.7; 5.3-4; 5.6-7
Prentice Hall – Middle Grades Mathematics C2 (1995)	1.5; 3.8; 3.10; 6.2-4; 6.7; 8.2; 11.5
Prentice Hall – Middle Grades Mathematics C3 (1995)	1.6; 3.1; 3.8; 4.7; 5.2; 6.2-3; 7.8-9
SFAW – Middle School Math C1, V1 (1999)	2.1; 2.4; 2.11
SFAW – Middle School Math C1, V2 (1999)	9.7
SFAW – Middle School Math C2, V1 (1999)	1.5; 2.1-3; 2.5; 3.5-6; 3.10
SFAW – Middle School Math C2, V2 (1999)	10.2-5
SFAW – Middle School Math C3, V1 (1999)	2.1; 2.7-9; 3.1-3; 4.1; 4.3
SFAW – Middle School Math C3, V2 (1999)	10.1; 10.4
SFAW: UCSMP – Transition Mathematics, Part 1 (1998)	1.1-2; 2.1-4; 2.8-9; 4.2-3; 4.7
SFAW: UCSMP – Transition Mathematics, Part 2 (1998)	13.5