

CLEMSON MIDDLE SCHOOL MATHEMATICS PROJECT

UNIT 4: PATTERNS AND FUNCTIONS

PROBLEM 1: SPREADING RUMORS

On October 1, Tara, an eighth-grade student, starts a rumor that Christina Aguilera will play a concert at school. She tells two students the rumor that day, with the instructions that each of these two students should repeat the rumor to two more students the next day, and each of these students is to repeat the rumor to two more students on the following day, and so on. In other words, on the first day, two new students know the rumor; on the second day, four more new students know the rumor; on the third day, eight more new students know the rumor; and so on.

- A. If there are 800 students in the school, estimate the number of days it would take until all students have heard the rumor.
- B. Create a table relating the number of days to the people who hear the rumor that day and to the total number of people who have heard the rumor on that day.
- C. Compare your estimate in Part A with the actual answer.
- D. Use exponents to represent the numbers in the second and third columns.
- E. Write a formula that represents the numbers in the second and third columns.
- F. If the pattern continues, how many people would hear the rumor on the 10th day?
On the 20th day? On the 30th day?
- G. If 1,024 people hear the rumor on the 10th day, on which day would 512 hear it?
- H. By which day would 200 people have heard the rumor? 2,000? 20,000? 200,000?
- I. Explore the problem graphically.

MATERIALS

Casio *Algebra FX 2.0* Graphing Calculator

EXTENSION

Work backwards to determine the meaning of a zero exponent and negative exponents.

REFERENCE: "Patterns and Functions," Curriculum and Evaluation Standards for School Mathematics Addenda Series, Grades 5-8, NCTM, 1996, p.6.

PATTERNS AND FUNCTIONS

ONE SOLUTION TO PROBLEM 1: SPREADING RUMORS

- A. If there are 800 students in the school, estimate the number of days it would take until all students have heard the rumor.**

Answers will vary, but students' responses should give the teacher insight into their sense of exponential functions. Most students will probably suggest that it takes far more days than it actually does.

- B. Create a table relating the number of days to the people who hear the rumor that day and to the total number of people who have heard the rumor on that day.**

A table for the first ten days might look like the following.

DAY	NEW STUDENTS	TOTAL STUDENTS
1	2	3
2	4	7
3	8	15
4	16	31
5	32	63
6	64	127
7	128	255
8	256	511
9	512	1,023
10	1,024	2,047
11	2,048	4,095
12	4,096	8,191

- C. Compare your estimate in Part A with the actual answer.**

Many students may be surprised to discover that, if the pattern established by the problem continues, all of the students in the school will have heard the rumor by the 9th day. During discussion, emphasize that even though each person is only required to tell the rumor to two people, she/he must tell it to people who have not already heard it. In reality, after only a few days it might be difficult to find two such people, especially if the search is restricted to people in the school.

PATTERNS AND FUNCTIONS

D. Use exponents to represent the numbers in the second and third columns.

This is a chance for students to appreciate exponential notation. For the total number of students, the teacher may need to lead students to see that the numbers are always one less than a power of 2.

DAY	NEW STUDENTS	TOTAL STUDENTS
1	$2 = 2^1$	$3 = 2^2 - 1$
2	$4 = 2 \cdot 2 = 2^2$	$7 = 2^3 - 1$
3	$8 = 2 \cdot 2 \cdot 2 = 2^3$	$15 = 2^4 - 1$
4	$16 = 2 \cdot 2 \cdot 2 \cdot 2 = 2^4$	$31 = 2^5 - 1$
5	$32 = 2^5$	$63 = 2^6 - 1$
6	$64 = 2^6$	$127 = 2^7 - 1$
7	$128 = 2^7$	$255 = 2^8 - 1$
8	$256 = 2^8$	$511 = 2^9 - 1$
9	$512 = 2^9$	$1,023 = 2^{10} - 1$
10	$1,024 = 2^{10}$	$2,047 = 2^{11} - 1$
11	$2,048 = 2^{11}$	$4,095 = 2^{12} - 1$
12	$4,096 = 2^{12}$	$8,191 = 2^{13} - 1$

E. Write a formula that represents the numbers in the second and third columns.

This too may give students some difficulty, especially with the third column. However, students should become accustomed to extending patterns until they can generalize to a variable. The time spent helping students discover these formulas would be well spent.

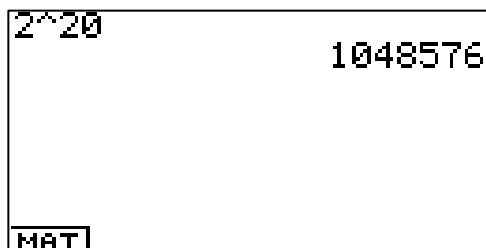
If we let x represent the number of the day, the second column can be represented by 2^x . The third column is $2^{x+1} - 1$.

F. If the pattern continues, how many people would hear the rumor on the 10th day? On the 20th day? On the 30th day?

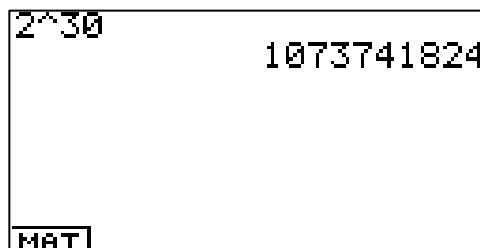
At this point, technology can play a significant role, one which students should appreciate. Because we have already determined an explicit formula in part D, we may wish to use the calculator to do the arithmetic. From the MAIN MENU, choose RUN-MAT, then simply:

PATTERNS AND FUNCTIONS

- x Type in 2, the caret \square^{\square} , which is used for exponents, 10, and press \square_{EXE} .
The result is 1,024 as shown in the table.
- x To edit the previous entry, press the left arrow key, until the 1 (in the 10) is highlighted, press \square_{DEL} , then type in 2, and press \square_{EXE} . See below left.
- x Press the left arrow key until the 2 in 20 is highlighted, press \square_{DEL} , then type in 3, and press \square_{EXE} . See below right.



A calculator display showing the calculation of 2 to the power of 20. The top line shows "2^20" and the bottom line shows the result "1048576". The "MAt" indicator is visible in the bottom left corner.



A calculator display showing the calculation of 2 to the power of 30. The top line shows "2^30" and the bottom line shows the result "1073741824". The "MAt" indicator is visible in the bottom left corner.

On the 10th day, 1,024 people would hear the rumor; on the 20th day, 1,048,576 would hear the rumor; and on the 30th day, 1,073,741,824 would hear it! Within a few more days the entire world would have heard it! From this, students should begin to develop a sense of how quickly exponential functions can grow (and of how quickly rumors can be spread).

G. If 1,024 people hear the rumor on the 10th day, on which day would 512 hear it?

The answer, of course, is on the 9th day. This example, and others similar to it, should also help students develop a sense of exponential (multiplicative) thinking; it's very different from the linear (additive) thinking to which they may have become accustomed. To facilitate this type of thinking, the teacher may wish to ask several questions involving the idea of half (the day before), one-fourth (two days before), and one-eighth (three days before). For example, the teacher might ask how many days after 4,096 people hear the rumor will 8,192 hear it.

PATTERNS AND FUNCTIONS

H. By which day would 200 people have heard the rumor? 2,000? 20,000? 200,000?

For this problem, a table would be extremely beneficial. From the MAIN MENU, choose GRPH-TBL. Then,

- x At the Y1: prompt, delete any function that is there by pressing **F2** followed by **EXE**. Then type in $2^{(X \div 1) + 1}$ and press **EXE**. See below left.
- x If necessary, press **F6** until RANG shows at the bottom of the screen. Press **F2** for range. You may choose to start at 0 and end at, say 30. The pitch should be 1, so you can view the total each day. Press **EXE** after typing in each of these values. See below right.

```

Graph Func :Y=
Y1:2^(X+1)-1
Y2:
Y3:
Y4:
Y5:
Y6:
SEL DEL TYPE GMEM DRAW
    
```

```

Table Range
X
Start:0
End :30
Pitch:1
    
```

- x Press **ESC** to return to Graph Func screen and **F5** to view the table.

Using the down arrow, you should be able to tell that on the 10th day, over 2,000 people have heard the rumor; by the 14th day, over 20,000 (actually 32,767) have heard it; and by the 17th day, 200,000 have heard it! See below for two of these values. Once again, students should begin to get a sense of how quickly exponential functions can grow.

X	Y1
9	1023
10	2047
11	4095
12	8191

10

X	Y1
14	32767
15	65535
16	131071
17	262143

17

PATTERNS AND FUNCTIONS

I. Explore the problem graphically.

We are really exploring two functions, one concerning the number of people who hear the rumor each day and one concerning the total number of people who have heard the rumor by a given day.

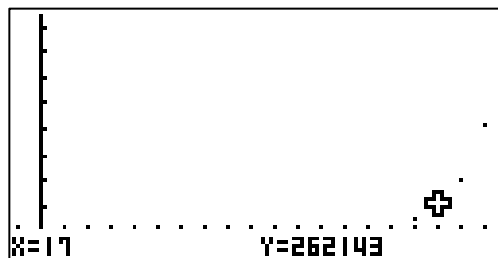
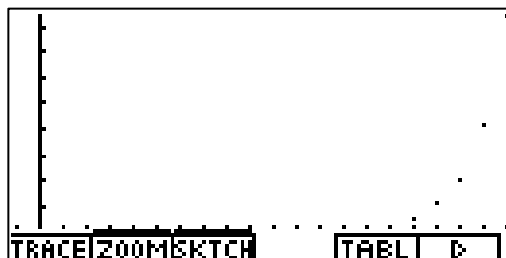
From the MAIN MENU, call up GRPH-TBL. If you have just completed part H, the function for the total number of people who have heard the rumor is already entered. (It may be necessary to press **ESC** until Graph Func appears at the top of the screen.

- x If the function is not there, type it in and press **EXE**.
- x The key to viewing and studying the graph is establishing an appropriate window. To see the window screen, press **SHIFT** **OPTN**.
- x Type in appropriate values, using the information from the table as a guide. For example, to look at the numbers for the first twenty days, you could set the Xmin at -1 (so the graph extends a little to the left of the desired values), the max at 20, and the scale at 1. Remember to press **EXE** after typing in each value. Use the down arrow key on the disc to bypass “dot.” You could set the Ymin at $-250,000$ (so the graph extends below 0), the max at, say 2,100,000, and the scale at 250,000. Note that $-250,000$ and 2,100,000 appear in scientific notation. See below. (To view the y-scale, simply scroll down.)

```
View Window
Xmin : -1
max  : 20
scale: 1
dot   : 0.16666666
Ymin : -2.5E+05
max   : 2.1E+06
INIT|TRIG|STD|STO|RCL
```

PATTERNS AND FUNCTIONS

- x Press **ESC** to return to the primary Graph Func screen and **F5** to return to the table. Now press **F5** G-PLT again to view a graph plot. See below left.
- x Pressing **F1** and the left and right arrow key on the disc allows you to trace through the values. The graph should reinforce for students how steep an exponential graph can become. See below right.



To look at the function representing the number of people who hear a rumor on a given day:

- x Return to the primary GRPH-TBL screen by pressing **ESC**. If you wish to look at only one function, de-select Y1 by pressing **F1** (for the select key) when the function is highlighted. This will cause the “=” to be “unhighlighted” or “unboxed.” If you wish to view both functions simultaneously, do not “de-select” it. (The same process “selects” the function.)

- x Use the down arrow key to the Y2 row and type in 2^X . Press **EXE**.

Notice the highlighted “=”.

If you are viewing only the function representing the number of people who hear the rumor on a given day, you may wish to reset the maximum value for Y. To do so, press **SHIFT** **OPTN** to obtain the window screen. Use the down arrow to scroll down for the maximum Y value and type in your new value. (One possibility is 1,100,000.) Press **EXE** to enter the value, **ESC** to leave the window-setting screen, **F5** to view the table, and **F5** to draw the graph. Note that both scatterplots seem to “hug” the x -axis for quite some time, but then become quite steep.

PATTERNS AND FUNCTIONS

PROBLEM 2: HEIGHT and LONG JUMP

Explore the relationship between the heights of the students in your class and the length of their standing long jump.

- A. Estimate the length of your standing long jump. Do you think there is a relationship between a person's height and the length of the long jump? If so, do you think the relationship is positive (taller people jump farther) or negative (shorter people jump farther)?
- B. Collect data from your entire class. Include each person's height, gender, and length of longest of jump.
- C. Are there any observable patterns? Is jump length related to height? Explore the relationship numerically, graphically, and algebraically.
- D. Is jump length related to height? Explain your thinking. Would you expect similar results if you gathered more data from students in your grade?
- E. What would you predict for students in different grades? Why?

MATERIALS

Tape measure or meter stick

Adhesive tape

Casio *Algebra FX 2.0* Graphing Calculator

EXTENSION

Separate the data for males from the data for females. Explore the relationship again. What similarities and differences do you see?

REFERENCE: "Measurement in the Middle Grades," Curriculum and Evaluation Standards for School Mathematics Addenda Series, Grades 5-8, NCTM, 1994, p.5.

PATTERNS AND FUNCTIONS

ONE SOLUTION TO PROBLEM 2: HEIGHT and LONG JUMP

- A. Estimate the length of your standing long jump. Do you think there is a relationship between a person's height and the length of the long jump? If so, do you think the relationship is positive (taller people jump farther) or negative (shorter people jump farther)?**

Answers will vary. Most students might expect that either there is no relationship or that there is a positive one in which taller people tend to jump farther. No matter their thoughts, encourage them to speculate.

- B. Collect data from your entire class. Include each person's height, gender, and length of longest of jump.**

Some discussion will be needed before data collection begins. How many jumps will each person get? How will the jumps be measured? How many people will do the measuring? These are important issues. Note that the question asks for the person's longest jump, not the average jump.

Data for a few students, which are provided in the "Measurement in the Middle Grades" addendum to the Standards, are shown below. Using actual data from your class will, however, be far more effective.

HEIGHT (in centimeters)	SEX	LENGTH (in centimeters)
110	F	100
160	F	120
151	M	123
126	M	135
100	F	91
125	M	121
141	F	108
162	M	128
132	F	102
143	M	115

PATTERNS AND FUNCTIONS

C. Are there any observable patterns? Is jump length related to height? Explore the relationship in numerically, graphically, and algebraically.

Answers will vary. For the data given, students may notice that the shortest person made the shortest jump, but the tallest person did not make the longest jump. It is difficult to tell whether much of a pattern or a relationship exists between these two variables when looking at only the raw, unsorted data.

The Casio *Algebra FX-2.0* provides an easy means of sorting the data. To begin, call up STAT from the MAIN MENU.

- x If data are entered into the lists that you don't need, highlight any value in the unwanted list, press **F6** if necessary so that DEL-A appears as one of the function choices. Press **F4** followed by **EXE** to delete the elements in the list. Move the cursor to List 2 and repeat the process. Alternatively, you could use Lists 3 and 4.
- x Type the heights into List 1, pressing **EXE** after each entry. Then type the jump lengths into List 2, again pressing **EXE** after each entry. Double check the data to be sure you have typed the numbers in correctly and that the numbers are paired correctly.
- x Use **F6** from the function keys if necessary so that SORT is shown as a function choice. Then press **F1** to select SORT, press 1 for SortA (sort in ascending order). The calculator asks how many lists you wish to sort. Type 2 and press **EXE**. The calculator asks which list should be the base. Type 1, indicating that List 1 is the base, and press **EXE**. Finally, the calculator asks you to select the second list. Type 2, indicating List 2 must be paired with values in List 1, and press **EXE**.

The data should now be sorted, perhaps allowing you to make more sense of the numbers. As you move the cursor down through List 2, look to see if there are any clear patterns. For the data presented here, none emerge, although there may be a

PATTERNS AND FUNCTIONS

small positive association. In other words, it appears that there may be a slight tendency for taller people to have slightly longer jumps. A key word here is slight; the relationship does not appear to be very strong.

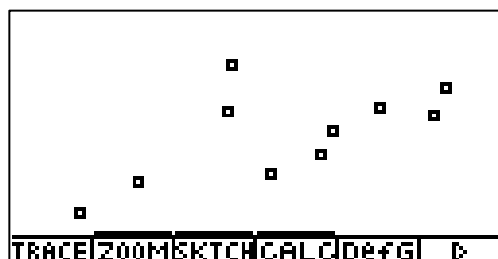
A scatterplot would be helpful to explore whatever relationship exists between these two variables. From the primary STAT screen,

- x Press **CTRL** **F3** to reach the set up for statistics. Make sure the S-Window is set to automatic. (If it is manual, then you will have to set the viewing window later on.) Press **ESC** when you are finished.
- x Press **F1** for graph. (Use **F6** if the GRPH option is not shown.) Press **5** for SET. From the StatGraph1 screen, highlight Graph Type and be sure it is a scatterplot by choosing Scat **F1**, with List 1 as the Xlist, List 2 as the Ylist, and 1 as the frequency. The function keys will be used to obtain each of these settings. You can also adjust the symbol used to mark the data points by scrolling down to Mark Type and selecting any one of the three options. Use the appropriate function keys and the down arrow key to move to the different choices. See below left. Press **EXE** after you have finished.
- x Again use **F6** so that GRPH is at the bottom of the screen if necessary. Press **F1**, then 1 for S-Gph1 to select graph 1. See below right.

```
StatGraph1
Graph Type  : Scatter
XList       : List1
YList       : List2
Frequency   : 1
Mark Type   : □

```

□	X	■
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PATTERNS AND FUNCTIONS

You should notice (at least for the data presented here) that the scatterplot reveals a fairly pronounced positive relationship. That is, the graph shows that taller people do have a tendency to make longer jumps. However, there are two data points that clearly stand apart from the others. To identify them:

- x Press **F1** , and then the right and left cursor arrow keys to trace the scatterplot. The two points that stand apart are (125, 121) and (126, 135). Nevertheless, the positive association is evident.
- x When you are finished tracing, press **ESC** to return to the scatterplot with other function options.

Several options are available to explore the relationship algebraically.

Pressing **F4** for CALC will give you options for various regressions. The least-squares regression line (2: Linear) and the median-median line (3: MedMed) can be examined from this command.

The least-squares regression line shows a correlation of 0.6227, a moderately strong relationship, with an equation of $y = .420x + 57.6$. See below left. The median-median line for the data presented here is $y = .46x + 49.2$. See below right.

```
LinearReg
a =0.42026666
b =57.564
r =0.62270726
r²=0.38776433
MSe=130.719966
y=ax+b
```

COPY/DEAL

```
Med-Med
a =0.46
b =49.17
y=ax+b
```

COPY/DEAL

At the middle-school stage, students are most likely not able to interpret slope and y -intercept if given an equation. Nevertheless, some exploration can help build their foundation for algebra. Give students a value for x and have them determine what the model suggests the y -value should be. For example, using the median-median line, someone who is 100 centimeters tall is predicted to have a jump length of $.46 \cdot 100 + 49.2 = 95.2$ centimeters. Note that the person in the class who was 100

PATTERNS AND FUNCTIONS

centimeters tall actually jumped 91 centimeters. In other words, this person did not jump as far as our model predicts.

A convenient way to do this is to create a list of predicted values, based on the regression formula.

- x Press **ESC** when viewing the regression graph to return to the primary statistics screen.
- x Use the right arrow key on the disc to move to List 3 and the up arrow key so that List 3 is highlighted.
- x Press .46, the multiplication key, **OPTN** **F1** , then **1** , and then **1** again.

At this point, you should have .46 times List 1 in the command area. Now simply add 49.2 to complete the formula. See below left for this formula.

- x Press **EXE** and the list of predicted values will be created. See below right.

	List 1	List 2	List 3	List 4
1	100	91		
2	110	100		
3	125	121		
4	126	135		
5	132	102		
.46×List 1+49.2				
LIST CPLX NUM PROB HYP ▷				

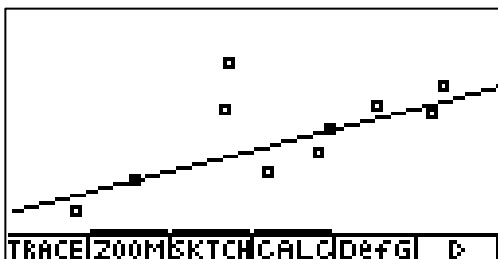
	List 1	List 2	List 3	List 4
1	100	91	95.2	
2	110	100	99.8	
3	125	121	106.7	
4	126	135	107.16	
5	132	102	109.92	
				95.2
GRAPH CALC TEST INTR DIST ▷				

Given this example, students should be able to determine how their actual jump length compares with the value the model predicts. Because some are above and others are below the predictions, this work with deviations – the difference between the actual and the predicted - could lead to work with positive and negatives.

Work with the deviations should also be connected to the graph. From the primary statistics screen, press **F1** for GRPH, then 1 for S-Gph1. Choose **F4** for CALC from the scatterplot screen, then 3 to calculate the median-median line. After the median-median line has been calculated, press **F6** to draw the regression line with the scatterplot. Again you can trace through the graph (press **F1** to do so), but this time note whether the points are on, above, or below the line. Data points that are on the prediction line have 0 deviation (the students jumped the distance the model

PATTERNS AND FUNCTIONS

predicted). Data points that are above the prediction line have positive deviations (the students jumped farther than expected). Data points that are below the prediction line have negative deviations (the students jumped less than expected). See below.



Students could look at all of the deviations. If desired, a list of deviation can be created. To create a list showing the difference between the actual jump length and the predicted jump length:

- x Press **ESC** when viewing the graph to return to the primary statistics screen. (You may need to press it twice.)
- x Use the right arrow key to move to List 4 and the up arrow key so that List 4 is highlighted.
- x Press **OPTN** **F1** **1** and then 2. You should see List 2 in the command area. Now press the subtraction key, **OPTN** , **F1** , then **1** for List, and 3 for List 3. See below right, which tells the calculator to take the difference between the actual value and the value the regression line expects.
- x Press **EXE** and the list of deviation should be created. See below right.

	List 1	List 2	List 3	List 4
1	100	91	95.2	
2	110	100	99.8	
3	125	121	106.7	
4	126	135	107.16	
5	132	102	109.92	
List 2-List 3				
LIST CPLX NUM PROB HYP				

	List 1	List 2	List 3	List 4
1	100	91	95.2	-4.2
2	110	100	99.8	0.2
3	125	121	106.7	14.3
4	126	135	107.16	27.84
5	132	102	109.92	-7.92
-4.2				
LIST CPLX NUM PROB HYP				

Students should practice interpreting the values. For example the first value, ? 4.2, tells us that the person who is 100 centimeters tall and jumped 91 centimeters, actually jumped 4.2 centimeters less than what might have been expected. In other words, negative deviations are represented by points below the line and indicate

PATTERNS AND FUNCTIONS

jumps below expectations, and positive deviations are represented by points above the line and indicate jumps above expectations.

Have your students find the mean deviation. If the least-squares line is used, it will be 0 because the positive deviations and negative deviations will balance out. If the median-median line is used, it will be close to, but not 0. For our problem, the mean deviation from the median-median line is approximately 3 centimeters. This idea may help students develop a deeper sense of both the mean and a regression line.

D. Is jump length related to height? Explain your thinking. Would you expect similar results if you gathered more data from students in your grade?

Students should recognize that from the data presented here, there is some evidence that jump length is related to height, that taller people have a tendency to be able to jump farther. Answers will vary as to whether or not more data will confirm the relationship already explored, but it is important for students to recognize two points:

- 1) Sampling is extremely powerful in looking for patterns and relationships.
- 2) Larger samples are preferable to smaller ones.

E. What would you predict for students in different grades? Why?

Answers will vary, but the idea of a positive relationship, with the length of the jump a function of the height, will probably remain.

PATTERNS AND FUNCTIONS

PROBLEM 3: THE RACE

Pat and his older sister, Terry, run a race. Pat runs at an average of 3 meters every second and Terry runs at an average of 5 meters every second. In a 100-meter race, Pat gets a 40-meter head start because he runs at a slower pace. Who wins the race? Provide a convincing argument.

EXTENSIONS

- 1) Create a graph of the two runners, using distance on the vertical axis and time on the horizontal axis. Write a paragraph analyzing the graph. Comment on the results of the race for different distances.
- 2) Discuss the slope (steepness) of the graph. Be as specific as possible.
- 3) Explore the race for different speeds and head starts.
- 4) Using times for a race that you and a friend have run, determine a head start and a distance that would result in a tie. Try it!

REFERENCE: "Patterns and Functions," Curriculum and Evaluation Standards for School Mathematics Addenda Series, Grades 5-8, NCTM, 1996, p.55.

PROBLEM 4: MAKING MONEY

You need to make some extra money and decide to do so by working in your neighborhood. You receive two job offers. In the first, your neighbor will pay you \$6 per hour to do whatever is needed. For the second offer, your neighbor will pay you \$20 per week plus \$2 for every hour you work.

Which job should you take? Be clear about your assumptions, and explore the problem with a table, a graph, and any other way you can imagine. Write a paragraph analyzing what you have found.

PATTERNS AND FUNCTIONS

TEXT SECTION CORRESPONDENCES

The materials in this module are compatible with the following sections in the listed texts.

TEXT	SECTION
AW – Foundations of Algebra and Geometry (1998)	1.2-3; 3.3; 4.1; 5.2; 6.1-2; 7.2; 8.1-3
Glencoe – Mathematics Applications and Connections C1 (1995)	1.4-5; 2.6; 3.10; 6.1; 8.4; 13.4; 13.5
Glencoe – Mathematics Applications and Connections C2 (1995)	3.7; 4.1-2; 6.4; 8.2; 14.3-5
Glencoe – Mathematics Applications and Connections C3 (1995)	2.6; 6.1; 7.5; 11.1-4; 13.4B
Houghton Mifflin – The Mathematics Experience I (1992)	4.9
Houghton Mifflin – The Mathematics Experience II (1992)	1.10; 3.11-12; 4.5; 5.13; 12.2-3; 15.5; 15.8
McDougal Littell – Gateways to Algebra and Geometry (1994)	1.1-3; 5.2-3; 8.2; 9.7; 10.7; 11.7; 12.2
Prentice Hall – Middle Grades Mathematics C1 (1995)	1.7; 5.1-2; 5.4-7
Prentice Hall – Middle Grades Mathematics C2 (1995)	
Prentice Hall – Middle Grades Mathematics C3 (1995)	1.6; 2.3; 4.7; 5.2-4; 6.2-3; 9.6
SFAW – Middle School Math C1, V1 (1999)	1.3; 2.9; 2.11; 5.1
SFAW – Middle School Math C1, V2 (1999)	9.7
SFAW – Middle School Math C2, V1 (1999)	1.5-7; 2.1; 2.3; 2.5
SFAW – Middle School Math C2, V2 (1999)	10.2-5
SFAW – Middle School Math C3, V1 (1999)	1.7; 3.1-3; 4.1; 4.3-5
SFAW – Middle School Math C3, V2 (1999)	
SFAW: UCSMP – Transition Mathematics, Part 1 (1998)	4.2-3; 4.7; 6.4-5
SFAW: UCSMP – Transition Mathematics, Part 2 (1998)	8.4; 13.1