

CLEMSON MIDDLE SCHOOL MATHEMATICS PROJECT

UNIT 5: GEOMETRIC RELATIONSHIPS

PROBLEM 1: PERIMETER AND AREA TRAINS

Let's define a train as the shape formed by congruent, regular polygons that share a side. Explore the relationship between the number of polygons and the perimeter for trains of triangles, quadrilaterals, pentagons, and hexagons. Continue the exploration until you discover a formula for the perimeter of a train with x cars. Generalize your results to a polygon with n sides. Finally, explore the graph of your formula for each of the shapes.

MATERIALS

Pattern blocks and/or color tiles

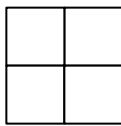
Casio *Algebra FX 2.0* Graphing Calculator

EXTENSIONS

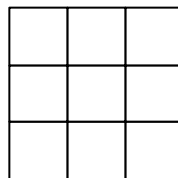
- 1) Form trains constructed with different polygons. For example, make a train with hexagons and triangles. Determine a pattern for the perimeter of the train as polygons are added.
- 2) Using squares, find a pattern for the perimeter and area of trains of different designs. For example, consider trains that make squares as started below.



Train 1



Train 2



Train 3

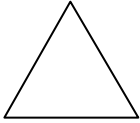
- 3) Make and explore trains in three-dimensions, such as those needed to stack boxes in a grocery store.

REFERENCE: "Patterns and Functions," Curriculum and Evaluation Standards for School Mathematics Addenda Series, Grades 5-8, NCTM, 1996, pp.49-54.

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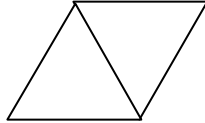
ONE SOLUTION TO PROBLEM 1: PERIMETER AND AREA TRAINS

Let's begin by exploring with triangles.



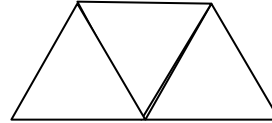
TRAIN 1

$$P = 3$$



TRAIN 2

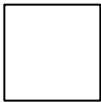
$$P = 4$$



TRAIN 3

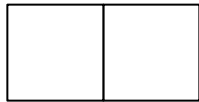
$$P = 5$$

Note that the perimeters increase by 1. Now we'll look at squares.



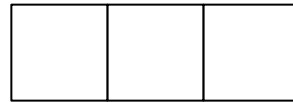
TRAIN 1

$$P = 4$$



TRAIN 2

$$P = 6$$



TRAIN 3

$$P = 8$$

Note that the perimeters increase by 2. What do you predict will happen to the perimeters of pentagonal trains?

After they have spent a significant amount of time exploring the problem using polygons with more and more sides, students may discover a relationship between the number of sides of the polygon and the rate of increase in the perimeter of the trains. Teachers should encourage their students to describe the patterns they have found.

Students should then try to generalize their discovery by trying to include variables. Depending upon the level and the sophistication of the students, teachers may need to lead the students a great deal at this point. Organizing their findings in a table, such as the one shown below, may help students make sense of this. The final cell in the table, the one in which the number of sides of the polygon and the number of cars in the train are both represented by variables, may be especially problematic for many students. Nevertheless, working on the problem until they can generalize their results can be extremely valuable in developing mathematical power.

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# of Sides	1 Car	2 Cars	3 Cars	x Cars
3	3	4	5	$3 + x + 1 + x + 2$
4	4	6	8	$4 + 2(x + 1) + 2x + 2$
5	5	8	11	$5 + 3(x + 1) + 3x + 2$
6	6	10	14	$6 + (x + 1) + 4x + 2$
n	n	$2n + 2$	$3n + 4$	$n + (x + 2)(x + 1) + x(n + 2) + 2$

We now wish to explore the graphs of our formulas for triangles, quadrilaterals, pentagons, and hexagons. If we let x represent the number of cars in a train and y represent the perimeter of the train, we have the following formulas:

?? Triangles: $y = x + 2$

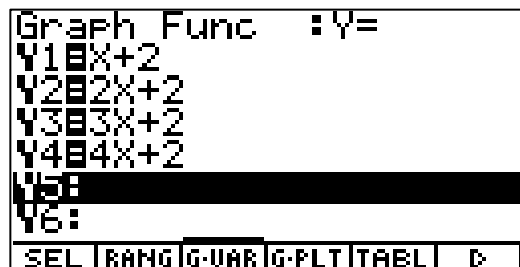
?? Quadrilaterals: $y = 2x + 2$

?? Pentagons: $y = 3x + 2$

?? Hexagons: $y = 4x + 2$

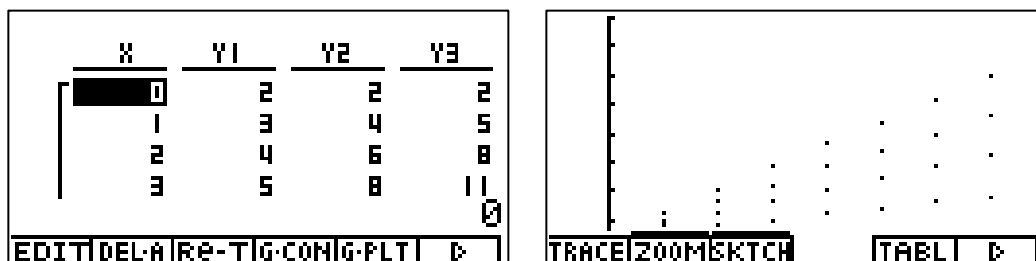
From the MAIN MENU, select GRPH-TBL.

- x Either delete any functions that are listed by highlighting them and pressing **F2** followed by **EXE**, or de-select them by highlighting them and pressing **F1**.
- x Type in the above formulas, pressing **EXE** after you complete each entry. Remember to use the **X,?,T** key to enter the x . See below.



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- x We'll now set a viewing window that will allow us to look closely at the graphs.
After you have entered the functions, press **SHIFT** **OPTN** for the viewing window. After you type in values for the following, press **EXE** . A possible X-Window is Xmin: -1, max: 8; scl: 1. Use the down arrow on the disc to bypass the "dot" and to get to the Y-Window values. Possible values Ymin: -2; max: 40; scl: 5. Press **ESC** when finished.
- x If necessary, press **F6** until TABL is an option. Press **F5** to see the table and **F5** again to view the graph. See below.



- x Press **F1** followed by the cursor movement keys (on the disc) to trace through the graphs. If the GraphFunc is turned on in the GRAPH SetUp (**CTRL** **F3**), the calculator will indicate which function you are tracing at any particular time. The vertical keys shift from one function to another, and the horizontal keys move along a single function.

Note that all of the graphs go through (0, 2). This would suggest that trains with 0 cars have a perimeter of 2! This, of course, does not make sense, but may help students begin to recognize the idea of domain; that is, we cannot let x represent just any number we want.

Students should note that all of the graphs become steeper as the number of sides in the polygons used to make up the trains increases. The triangles increase by one for each additional car; this means as we move to the right one unit, we must move up one unit. The quadrilaterals increase by two for each; in other words, on the graph as we move to the right one unit, we must move up two units. For the pentagons, for each move

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to the right of one unit we move up three units. Finally, for the hexagons, for each unit we move to the right on the graph we must move up four units.

Students should explore this idea, comparing the steepness of the graph with the numerical results we looked at earlier in the problem.

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PROBLEM 2: STRETCH THAT PICTURE

Suppose you have made an ink drawing that your family would like to frame and hang in your home. The drawing would be great in any of several places in your home, so your family is considering using a copy machine to either enlarge or reduce the size of the picture. Within the drawing are many shapes. In this problem you will consider what happens to the area and to the perimeter (for the framing) as you change the size.

- A. On graph paper, draw a square one unit on each side. Draw a second square that doubles each side. Draw a third square that triples each side of the original. Find the perimeter and area of the original and of the two larger squares.
- B. Draw a square that is four units on each side. Draw three more squares, one that doubles each side, one that triples each side, and one that halves each side. Compare the perimeters and areas of the three squares with the original. Make a conjecture as to what happens to the perimeter and area of the square.
- C. Again using graph paper, try stretching each side of different shapes by different factors and explore what happens to the perimeter and area. Be sure to include circles in your exploration.
- D. Using your calculator, explore what happens to the perimeter and area of a square when you double each side.
- E. Again using your calculator, explore what happens to the circumference and area of a circle when its radius is doubled.

MATERIALS

Casio *Algebra FX 2.0* Graphing Calculator

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ONE SOLUTION TO PROBLEM 2: STRETCH THAT PICTURE

- A. On graph paper, draw a square one unit on each side. Draw a second square that doubles each side. Draw a third square that triples each side of the original. Find the perimeter and area of the original and of the two larger squares.**

After students draw this on graph paper, they should discover the following results.

LENGTH OF SIDE	PERIMETER	AREA
1	4	1
2	8	4
3	12	9

Note that doubling a side doubles the perimeter and tripling a side triples the perimeter. Also, doubling a side quadruples the area and tripling a side multiplies the area by 9.

- B. Draw a square that is four units on each side. Draw three more squares, one that doubles each side, one that triples each side, and one that halves each side. Compare the perimeters and areas of the three squares with the original. Make a conjecture as to what happens to the perimeter and area of the square.**

LENGTH OF SIDE	PERIMETER	AREA
4	16	16
8	32	64
12	48	144
2	8	4

As before, the perimeter is multiplied by the same factor each side is multiplied by. In other words, if we double a side, we double the perimeter; if we multiply a side by 5, it appears that we would multiply the perimeter by 5.

For area, however, the relationship is more complicated. If we multiply each side by 2, the area is multiplied by 4, which we may observe is 2×2 . If we multiply

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each side by 3, the area is multiplied by 9, which we note is 3×3 . If we cut each side in half, the area is one-fourth the original, which is $\frac{1}{2} \times \frac{1}{2}$.

C. Again using graph paper, try stretching each side of different shapes by different factors and explore what happens to the perimeter and area. Be sure to include circles in your exploration.

Students should have adequate time to explore this. If possible, you may wish to enlarge and reduce pictures on a copy machine and measure the results. The goal is for students to discover and then convince themselves that as we stretch any shape by a factor of a , we stretch the perimeter by a factor of a and the area by a factor of a^2 .

D. Using your calculator, explore numerically and graphically what happens to the perimeter and area of a square when you double each side.

Now that students have developed a visual sense of what is occurring as figures are stretched (or shrunk) to create similar figures, we'll explore what is occurring numerically and graphically. Although we are looking here at what happens when we double each side, the techniques shown here can be used when we stretch the figure by any factor.

Let's begin by creating a list of values that will represent different possibilities for the sides of a square. From the MAIN MENU, choose RUN-MAT. Then, we'll create a sequence of values that can represent the sides of different squares.

- x Press **OPTN**, **F1** for LIST and **3** for Sequence. To put the counting numbers 1 through 10 into List 1, type in X (using the **X,?,T**), a comma, X (again using the same key), a comma, 1 (for the start), 10 (for the end), a comma, 1 so that the numbers count by 1's, the right parenthesis, the store key (the right arrow above **AC/ON**), then **F1** for List and **1** for List, **1** again for List 1, then **EXE**. See below left.
- x We will double these numbers for List 2, representing the sides of our enlarged squares. Type in 2, the multiplication sign, **F1** then **1** for List and

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the area of the squares, all we need do is multiply the numbers in the lists by themselves.

x Press $\boxed{\text{F1}}$ then $\boxed{1}$ for List, type 1 for List 1, the multiplication key, $\boxed{\text{F1}}$ then $\boxed{1}$ for List, type 1 for List 1, the store key, $\boxed{\text{F1}}$ then $\boxed{1}$ for List, type 5 for List 5, and press $\boxed{\text{EXE}}$. See below left. (Alternatively, we could use the X^2 key, typing in $(\text{List } 1)^2$.)

x Press $\boxed{\text{F1}}$ then $\boxed{1}$ for List, type 2 for List 2, the multiplication key, $\boxed{\text{F1}}$ then $\boxed{1}$ for List, type 2 for List 2, the store key, $\boxed{\text{F1}}$ then $\boxed{1}$ for List, type 6 for List 6, and press $\boxed{\text{EXE}}$. See below right.

List 1×List 1→List 5	Done						
<table border="1" style="width: 100%; border-collapse: collapse; text-align: center;"> <tr> <td style="padding: 2px;">LIST</td> <td style="padding: 2px;">MAT</td> <td style="padding: 2px;">CPLX</td> <td style="padding: 2px;">CALC</td> <td style="padding: 2px;">NUM</td> <td style="padding: 2px;">▶</td> </tr> </table>		LIST	MAT	CPLX	CALC	NUM	▶
LIST	MAT	CPLX	CALC	NUM	▶		

List 1×List 1→List 5	Done						
List 2×List 2→List 6	Done						
<table border="1" style="width: 100%; border-collapse: collapse; text-align: center;"> <tr> <td style="padding: 2px;">LIST</td> <td style="padding: 2px;">MAT</td> <td style="padding: 2px;">CPLX</td> <td style="padding: 2px;">CALC</td> <td style="padding: 2px;">NUM</td> <td style="padding: 2px;">▶</td> </tr> </table>		LIST	MAT	CPLX	CALC	NUM	▶
LIST	MAT	CPLX	CALC	NUM	▶		

Our goal is now to compare the perimeters and the areas. From the MAIN MENU, go to STAT. The first four lists are visible. Use the down and up arrows on the disc as desired. Note that the numbers in List 4 are always double those in List 3. In other words, the perimeter of the squares doubles when the sides are doubled. See below left for the beginnings of the first four lists.

Now use the right arrow key so that Lists 5 and 6 are visible simultaneously. Again, use the up and down cursor keys to move through the lists. Note that the numbers in List 6 are four times the numbers in List 5. In other words, when each side is doubled, the area is quadrupled. See below right for the beginning of the lists.

	List 1	List 2	List 3	List 4						
1	1	2	4	8						
2	2	4	8	16						
3	3	6	12	24						
4	4	8	16	32						
5	5	10	20	40						
<table border="1" style="width: 100%; border-collapse: collapse; text-align: center;"> <tr> <td style="padding: 2px;">GRAPH</td> <td style="padding: 2px;">CALC</td> <td style="padding: 2px;">TEST</td> <td style="padding: 2px;">INTR</td> <td style="padding: 2px;">DIST</td> <td style="padding: 2px;">▶</td> </tr> </table>					GRAPH	CALC	TEST	INTR	DIST	▶
GRAPH	CALC	TEST	INTR	DIST	▶					

	List 3	List 4	List 5	List 6						
1	4	8	1	4						
2	8	16	4	16						
3	12	24	9	36						
4	16	32	16	64						
5	20	40	25	100						
<table border="1" style="width: 100%; border-collapse: collapse; text-align: center;"> <tr> <td style="padding: 2px;">GRAPH</td> <td style="padding: 2px;">CALC</td> <td style="padding: 2px;">TEST</td> <td style="padding: 2px;">INTR</td> <td style="padding: 2px;">DIST</td> <td style="padding: 2px;">▶</td> </tr> </table>					GRAPH	CALC	TEST	INTR	DIST	▶
GRAPH	CALC	TEST	INTR	DIST	▶					

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Students may wish to explore the problem graphically. One possibility is to compare scatterplots. To compare the perimeters, we could look at a scatterplot based on (List 1, List 3) with one based on (List 1, List 4). We should see that the second increases at twice the rate of the first. We could compare areas by comparing the scatterplot from (List 1, List 5) with the one from (List 1, List 6). Details of this investigation are left to the reader.

Another possible method of investigating the relationship graphically is through functions. First, let's look at the perimeter. We will identify our variables as follows:

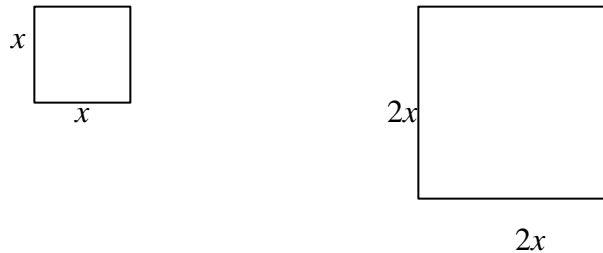
?? Let x represent the length of a side of the original square in units.

?? Let y represent the perimeter of the square in the same units.

First, let's note that if we double each side of the square, the length of each side becomes $2x$.

The perimeter of the original square can be described by the function $y_1 = x + x + x + x = 4x$. In other words, for any square with side length x , the perimeter is simply $4x$.

For the stretched square, the perimeter can be described by the function $y_2 = 2x + 2x + 2x + 2x = 4(2x) = 8x$. In other words, if the length of a side in the original square is x units, the perimeter of the stretched square is 8 times this. A picture of our two squares is shown below.



We could look at our two functions, $y_1 = 4x$ and $y_2 = 4(2x)$ using the GRPH-TBL function on the calculator. (The expression $4(2x)$ may be easier for students to understand than the simplified form $8x$.)

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We could look at the area in a similar way. Let y represent the area of the squares. The areas of our two squares can be represented by the functions $y_1 = x \cdot x = x^2$ and $y_2 = (2x) \cdot (2x) = (2x)^2 = 4x^2$. Our goal here is not the simplification of the algebraic expression, although that in itself can be a very useful exploration. The form that we could use for the second function, in the hope that it will be both clear and simple for students, is $y_2 = (2x)^2$. If desired, these two functions could be compared using the GRPH-TBL option from MAIN MENU.

E. Again using your calculator, explore numerically and graphically what happens to the circumference and area of a circle when its radius is doubled.

We can keep Lists 1 and 2 the same, but this time we will let them represent the radii of various circles. In List 2, we have doubled the values that appear in List 1. We will use a similar technique. From the MAIN MENU, choose RUN-MAT.

- x Press **OPTN** **F1** then **1** so that List is available.
- x To find the circumference of a circle, we want to double the radius and multiply it by Pi. To do so, type 1 behind the List on the screen to indicate List 1, the multiplication key, then type 2 (to double entries from List 1), multiplication key, **SHIFT** **EXP** for π , the store key, **F1** then **1** for List, type 3 for List 3, and **EXE**. See below left. This will put the circumferences of the original circles into List 3.
- x We'll now put the circumferences of the enlarged circles into List 4. Press **F1** then **1** for List, 2 for List 2, the multiplication key, then type 2, multiplication key, then **SHIFT** **EXP** for π , the store key, **F1** then **1** List, then type 4 for List 4, and **EXE**. See below right.

List 1 x 2 x π \rightarrow List 3
Done

LIST MAT CPLX CALC NUM D

List 2 x 2 x π \rightarrow List 4
Done

LIST MAT CPLX CALC NUM D

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The area formula for a circle is Pi times the square of the radius. We will now enter the areas of the original circles into List 5 and the areas of the larger circles into List 6.

x Press **SHIFT** **EXP** for π , the multiplication key, **F1** then **1** for List, 1 for List 1, the X^2 key, the store key, **F1** then **1** for List, 5 for List 5, and **EXE**. See below left.

x We'll now take an easier route for the second set of areas. Simply use the disc cursor keys to move the cursor to the left. Change the 5 to a 6 and the 1 to a 2. (The **DEL** key must be used to delete the 5 and 1.) Then press **EXE**. See below right.

```
List 1×2×π→List 3
                        Done
List 2×2×π→List 4
                        Done
π×List 1²→List 5
                        Done
LIST|MAT|CPLX|CALC|NUM|D|
```

```
π×List 2²→List 6
LIST|MAT|CPLX|CALC|NUM|D|
```

As before, go to the STAT function from the MAIN MENU and compare Lists 3 and 4. You should note that, once again, List 4 is double List 3. In other words, doubling the radius of a circle doubles the circumference. Also, List 6 is once again four times List 5, indicating that doubling the radius quadrupled the area.

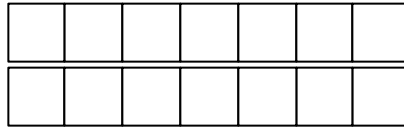
Many similar questions, involving different shapes and different factors to multiply by, arise. This type of exploration is appropriate for investigating many such problems.

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PROBLEM 3: TABLES FOR A PARTY

To make extra money, you decide to set up birthday parties for small children. Your neighbors have small, square card tables that they will lend you so you can set up refreshments. Unfortunately, the tables are difficult to move, so you want to borrow as few as possible. So that all of the children can sit together, you will place the card tables together into rectangles. Only one child can sit on each side of one of the tables.

- A. If your first party has 18 children, determine the dimensions of all possible rectangles that will seat all of the children. Organize the information into a table in which you have four columns: the bottom edge in units, the side edge in units, the perimeter in units, and the area in square units. For example, one arrangement, pictured below, has a bottom edge of 7 units, a side edge of 2 units, a perimeter of 18 units (all arrangements should have a perimeter of 18 units, one for each child), and an area of 14 square units (it takes 14 tables for this arrangement).



- B. What is the fewest number of tables you will need?
- C. What is the greatest number of tables you could use?
- D. Construct a scatterplot using number of tables on the bottom edge as one variable (perhaps in List 1) and the number of tables for the side edge as the other (perhaps in List 2). What shape do you get? Find the equation that best fits the scatterplot. Comment on what you have found.
- E. Construct a scatterplot using the number of tables on the bottom edge as one variable and the area as the other variable. This shape is called a parabola. To find the equation of the best fitting parabola, perform x^2 regression. What is the equation?
- F. If we do not restrict our problem to whole numbers, is there a rectangle with perimeter 18 that has the largest possible area? If so, find the length, the width, and the area.

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EXTENSIONS

- 1) For any given perimeter (in the problem, this was 18), predict which rectangle will have the maximum area. Support your answer with several examples.
- 2) Find a formula that will find the area of a rectangle for any given perimeter.
- 3) Explore the problem again, but use rectangular tables that can seat two children on the length and one child on the width.

REFERENCE: "Patterns and Functions," Curriculum and Evaluation Standards for School Mathematics Addenda Series, Grades 5-8, NCTM, 1996, pp.41-46.

GEOMETRIC RELATIONSHIPS

PROBLEM 4: CIRCLES, CIRCLES, CIRCLES

Find five circular objects in or around the room. Have each member of your group measure the distance across the circles (through the center) and around the circles.

Determine a single, agreed upon measure for both the distance across and the distance around each circle.

- A. Complete the chart below.

CIRCULAR OBJECT	DISTANCE ACROSS	DISTANCE AROUND

- B. Construct a scatterplot, using the distance around as the dependent variable (the vertical axis).
- C. Calculate the best-fit regression line for your data.
- D. How well does the regression line fit the data? How can you tell? Be specific.
- E. Using your regression line as a mathematical model for the relationship between the distance across a circle and the distance around it, what would the distance around a circle be if the distance across it is 0? (Note: this value is called the y -intercept.) What do you believe it really should be? How do you account for this difference?
- F. For what values of x (the domain – the distance across the circles), would you expect your model to work? Explain.
- G. Again using your regression line as a mathematical model, create a table of values showing the distance around a circle if the distance across the circle is each of the following: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.
- H. Use your table to answer this question. Each time the distance across the circle increases by one, by how much does the distance around the circle change? Does this happen every time? (Note: this is called the slope.)

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- I. Now use this idea of slope to answer the following. If a plane flies around the earth at the equator at an altitude of 10 miles, how much farther does it fly than it would if it traveled along the ground? (Hint 1: If the altitude is 10 miles, the distance across the equator, which is a circle, increases by 20 miles. Hint 2: The distance around the earth at the equator is about 25,000 miles, but you don't need to know this to solve the problem.)
- J. Is there a connection between your regression model and something you have studied earlier in a math class? If so, what? Discuss this, including any differences you find between this approach and what you studied earlier.

EXTENSIONS

Use the measures determined by each individual to determine the mean for each of the measures for all of the circles. Look at the differences from the mean for each measure. Then address the following questions. Be sure to support your answers with mathematical arguments.

- 1) Does anyone in the group consistently measure less than the group mean?
- 2) Does anyone consistently measure more than the group mean?
- 3) Who in the group measures consistently closest to the group mean?
- 4) Who in the group measures consistently far away from the group mean?

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TEXT SECTION CORRESPONDENCES

The materials in this module are compatible with the following sections in the listed texts.

TEXT	SECTION
AW – Foundations of Algebra and Geometry (1998)	1.2-3; 2.3; 3.3; 4.1; 5.2; 8.1-2
Glencoe – Mathematics Applications and Connections C1 (1995)	2.6; 3.10; 4.2; 4.9; 13.5
Glencoe – Mathematics Applications and Connections C2 (1995)	3.7; 5.6; 6.4; 6.7; 9.10; 14.3-4
Houghton Mifflin – The Mathematics Experience I (1992)	8.11; 8.13
Houghton Mifflin – The Mathematics Experience II (1992)	1.2-4; 1.6; 1.10; 3.8; 3.10; 4.5; 12.2-3
McDougal Littell – Gateways to Algebra and Geometry (1994)	5.3; 9.4; 11.7; 12.4
Prentice Hall – Middle Grades Mathematics C1 (1995)	1.7; 5.4; 5.6-7; 6.2; 6.4; 6.8
Prentice Hall – Middle Grades Mathematics C2 (1995)	1.5; 3.10; 5.2; 5.4; 6.7; 11.2
Prentice Hall – Middle Grades Mathematics C3 (1995)	1.6; 2.2; 4.7; 5.2-3; 6.2-3; 6.7
SFAW – Middle School Math C1, V1 (1999)	1.3; 2.11; 4.1; 4.4
SFAW – Middle School Math C1, V2 (1999)	9.7
SFAW – Middle School Math C2, V1 (1999)	1.5-6; 2.1; 2.3; 2.5; 5.5
SFAW – Middle School Math C2, V2 (1999)	9.1; 10.2-5; 11.6
SFAW – Middle School Math C3, V1 (1999)	1.7; 3.1-3; 4.1; 4.3; 4.5
SFAW – Middle School Math C3, V2 (1999)	9.1; 9.3; 10.1-2
SFAW: UCSMP – Transition Mathematics, Part 1 (1998)	3.8; 4.2-3; 4.7
SFAW: UCSMP – Transition Mathematics, Part 2 (1998)	8.4; 11.3