CLEMSON ALGEBRA PROJECT
UNIT 12: EXPONENTIAL AND LOGARITHMIC MODELS

PROBLEM 1: MODELING A BOUNCING BALL

You are to drop a “bouncy” ball onto a hard, level floor and explore the relationship between the maximum height of the bounce and how many times it has bounced.

A. Determine and calculate an appropriate model.

B. Investigate the errors in the model.

C. Use the model to predict the height of the 10^{th} and 100^{th} bounces. According to the model, does the ball stop bouncing? Why or why not?

D. Use the model to predict how many bounces it will take for the ball to reach heights of 5 and .01 cm. Show at least two methods of finding the solutions.

MATERIALS

Casio CFX-9850Ga PLUS or ALGEBRA FX2.0 Graphing Calculator

A “bouncy” ball

Meter Stick or Tape Measure

Digital Camera: QV-780 (optional)

EXTENSIONS

1. How does the model change if you measure down to instead of up from the floor?
2. How will the model change if you drop the ball from 200 cm?
3. Explore changes in the model by using different balls.
4. What changes in the model occur if the ball bounces sideways?

ONE SOLUTION TO PROBLEM 1: MODELING A BOUNCING BALL

A. Determine and calculate an appropriate model.

1) Make a conjecture as to which model is appropriate. Discuss the pros and cons.

When an object bounces, it rebounds a certain percentage of the height from which it was dropped, a value called the coefficient of restitution. Because this percentage should remain constant, the appropriate model is exponential. This is the case whenever for consistent additive changes in the independent variable, in this case the number of bounces, there are consistent multiplicative changes in the dependent variable, in this case the maximum height of the bounce.

Students may have difficulty determining the model and may wish to look at a scatterplot before making a conjecture as to the type of model to use. This is a natural and good intuitive approach. However, when they calculate a regression model, caution them that without a rationale for a particular model, they should conduct several experiments with as many data points as possible to validate their choice. Without this confirmation, the model they have chosen may fit their particular sample data, but may not generalize to other similar situations.

Similarly, even though their calculators can perform very complex regressions, such as quartic, higher degree models require both a theoretical justification and an abundance of data points to allow for generalizations and predictions, two primary purposes of regression. To convince yourself how well complex models can fit any data, make up, say, six points at random. Calculate the quartic regression model and observe how well it fits your data. With only a few data points, complex models are too sample specific to be of much use in the real world. They should not be generalized without a supporting theory.

For this problem, stress the idea of a multiplier effect on the dependent variable. For example, if the coefficient of restitution is 70%, every time the ball bounces it returns to 70% of its previous height. Help students develop an intuitive feel for this. Certainly the relationship cannot be linear, for if a ball is dropped from 20 centimeters and bounces back to a height of 10 centimeters, it will not bounce back 0 centimeters on the second bounce and negative 10 centimeters on the third!

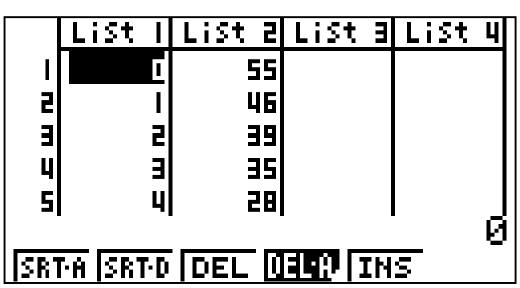
2) Gather the data.

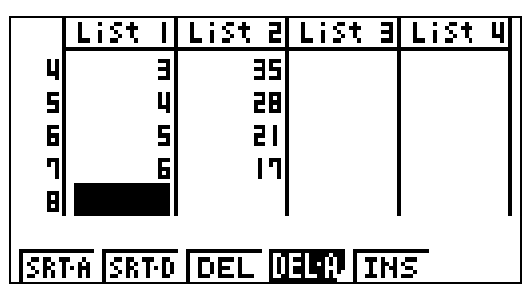
In this section, a simple method of gathering data is described. At the end of the section, a description of a method to gather data using a digital camera is also provided. For the experiment conducted in the development of the unit, the seven data points are shown on the two screens below.

Work in groups of at least three. Find a wall which allows your group ample room. Attach a tape measure to a wall, measuring up from the floor for perhaps a meter. Practice dropping the ball a few times so that you can get a few “clean” bounces that are reasonably vertical. Then drop the ball from a comfortable height, perhaps less than a meter. Have at least one partner “eyeball” the maximum height the ball reaches on each bounce, saying this height aloud so that another member of the group can record the results. This may take a few attempts. Then, to enter the data into your calculator:

- From the MAIN MENU, choose “List.”
- If data that you do not need are already in Lists 1 and 2, press [F4] to delete all elements of the list and [F1] to confirm the deletion. If you would prefer, you can use different lists or even different files. To access different files, from the LIST menu press [SHIFT] [MENU] to obtain the SET UP, and then the appropriate function key for a different file. Press [EXIT] to return to the list. (You could also use different lists.)
- Type the data into the two lists, pressing [EXE] after each entry.

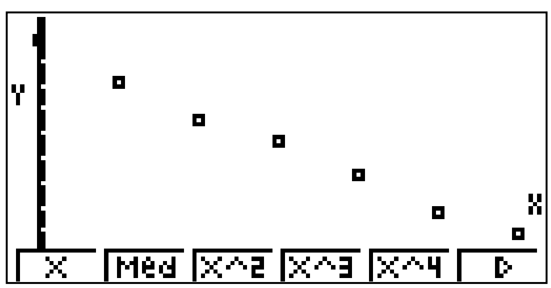
The results from one trial are shown below. List 1 shows the number of the bounce, and List 2 the maximum height of the ball, measured in centimeters.





3) Construct a scatterplot, setting an appropriate window.

- Go to the primary "Statistics" screen. You should see the data lists.
- Press [F1] to go to the GRPH menu, then [F6] for SET.
- Set StatGraph1 as a Scatter graph, using List1 for the Xlist and List2 for the Ylist. The Frequency should be 1, and the Mark Type and Graph Color (for the 9850Ga Plus) can be whatever you desire. Press [EXIT] twice to return to the primary Statistics screen.
- Press [SHIFT] [MENU] to check the SET UP for "Statistics." Make sure the Stat Wind is set to "Automatic" so the calculator will use the data to establish an appropriate range for both *x* and *y*. You will probably also want the Axes On. Move down to "Resid List" and, press [F2] for LIST and [F3] for List 3. This will put the residuals (discussed later) automatically in List 3 when regression is performed. Once you are sure the SET UP for STAT is what you want, press [EXIT] to return to the primary "Statistics" screen.
- To view the scatterplot, press [F1] for GRPH, then [F1] again to obtain GPH1. The scatterplot for the listed data is shown below.



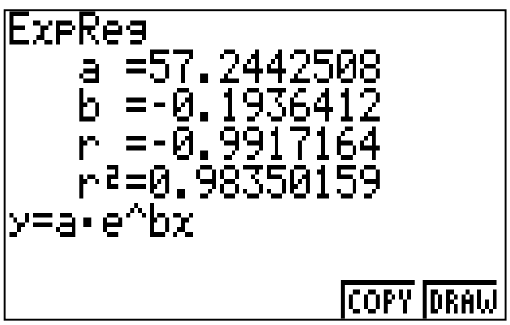
- By pressing [SHIFT] [F1] you can access the Trace function. Use the right and left arrows to see the values of each of the data points. Pressing [EXIT] and [F1] will return you to the graph without the Trace function.

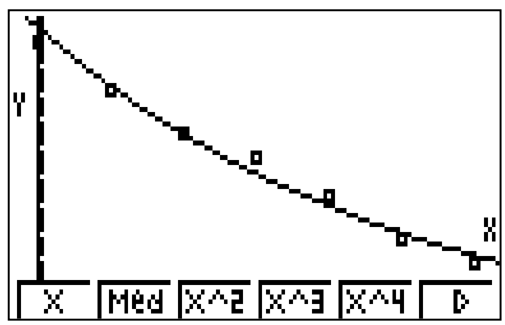
Note that the graph above does look reasonably linear. The earlier discussion concerning the type of model that is appropriate becomes extremely relevant here and should reinforce for students that regression analysis is not something that should be done merely by pressing buttons. Reasoning and analysis are, if anything, more important than ever.

4) Calculate the regression model and graph the regression model on the scatterplot.

x With the scatterplot in view, the bottom menu gives several regression models. Because we are interested in an exponential model, press [F6] for more choices and [F2] for the exponential model.

x Pressing [F5] and [EXE] will copy the regression equation into Y1. Pressing [F6] will draw the regression equation on your scatterplot, giving a visual representation of how well the mathematical model fits the data. This is shown in the figure on the right below.





5) Interpret the regression equation.

The regression equation is $y = 57.24 * e \wedge (-0.1936x)$. The *y*-intercept of our function is 57.24. Thus, 57.24 refers to the predicted height of the ball, in centimeters, after 0 bounces. In our problem, we know the ball was dropped from a height of 55 centimeters, so the error between our known value and predicted value is –2.24 cm; that is, our actual data point is 2.24 cm below the value predicted by our equation.

Casio calculators use $y = a * e \wedge (bx)$ form for exponentials, more in keeping with what is used in the "real world" than the $y = a * b \wedge x$ form used in most secondary texts. This might be a good opportunity to review the rule for exponents that states that $(a \wedge b) \wedge c$ is equivalent to $a \wedge (bc)$. Because $e \wedge -0.1936 = 0.8240$, we could write our regression equation as $y = 57.24 * .8240 \wedge x$. The .8240 represents the coefficient of restitution; it is the percentage of the height the ball will bounce back on this particular surface.

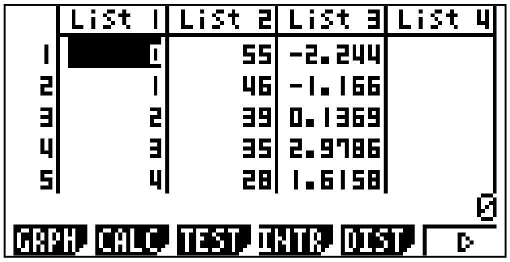
B. Investigate the errors in the model.

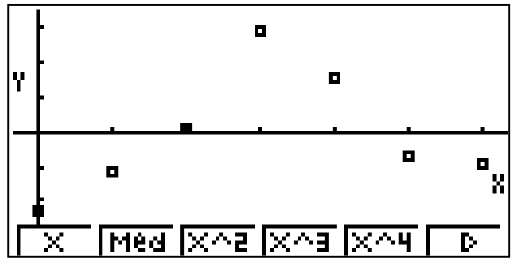
6) Calculate the residuals, and make a scatterplot of residuals to determine if the model is appropriate. Make conjectures as to underlying reasons for the residuals.

The errors, or residuals, in the model are the differences between the observed *y*-values (the position on the meter stick) and the *y*-values predicted by the regression equation. The calculator can create a list of residuals easily. Because of the SET UP work done earlier, the residuals have already been created for us and are stored in List 3.

The screen shown below on the left shows the beginning of this residual list. A scatterplot of these residuals is often valuable. Unlike much in the world of mathematics, our hope is that there is no pattern to the residuals! If there is one, then our model is not the most appropriate for the data. Below right shows the scatterplot of residuals, using List 1 as the *x*-value and List 3 as the *y*-value. To obtain this graph, begin at the primary "Statistics" screen. Then,

- Press [F1] for the GRPH menu, [F6] to SET the graph, and [F2] to work on GPH2.
- Make the appropriate choices for the scatterplot, and press [F2] to see the graph.



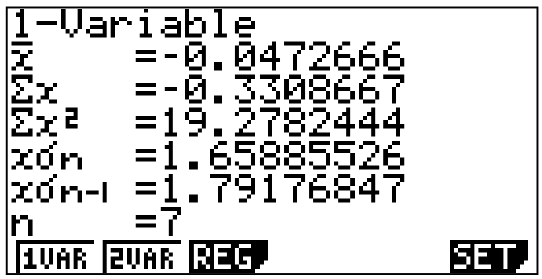


The residual plot does not appear to be as random as we would like. All of our positive residuals are in the middle of the graph, while the negative residuals are at both ends. In fact, the graph looks like it could be modeled with a reasonably smooth curve. This might indicate to us that our choice of models was not appropriate or that our data are not very accurate. Perhaps the biggest problem is that we have so few data points upon which our model is based. Although we may be convinced that an exponential model is the best, this example demonstrates that our mathematical models are not always as efficient as we would like. Nevertheless, we might also notice the absolute size of the errors in our model; the largest error is less than 3 centimeters. Perhaps, after all, our model is really not so bad.

Many causes for errors in our regression model are likely. First and foremost, "eyeballing" the maximum height is not a precise method of determining the maximum height. Further, the object may not have been perfectly spherical, causing disruptions in its bounces. The floor itself may not have been smooth, level, or even consistent in texture. Considering these likely sources of error, our model is remarkably good! Nevertheless, as with all regression, students who are accustomed to thinking of mathematics as unambiguous and even perfect may have difficulty accepting the idea that a mathematical model can be flawed.

7) Calculate the mean and sum of the residuals.

- From the primary "Statistics" screen, choose [F2] for "Calculate," [F6] to SET the calculations, and [F3] so that one-variable statistics are performed on List 3.
- Press [EXIT] to back up to the "Calculate" screen, followed by [F1] to perform the one-variable calculations. The screen below shows the results.



Note that the mean of the residuals, denoted by $\bar{x}$, is only –0.04. Although this value is quite small, it is not extremely close to 0, unlike the mean of errors for linear regression. Similarly, the sum of the residuals, denoted by $\sum x$, though small, is no longer 0.

C. Use the model to predict the height of the 10th and 100th bounces. According to the model, does the ball stop bouncing? Why or why not?

8) Substitute 10 and 100 for x into the regression equation.

Certainly one way to determine the values of $f(10)$ and $f(100)$ is to plug 10 and 100 in for *x* and work the problem numerically. The calculator makes finding function values very simple.

- From the "Run" menu, press [VARS] for the variables menu, followed by [F4] for GRPH.
- Type 10. Then press [→] for store, [X,θ,T] for the *x*-variable, and [EXE]. This will store 10 in for *x*.

- Press [F1] for the Y variable, followed by 1, so that Y1 is displayed. (Recall that our regression function is stored in Y1.)
- Upon pressing [EXE] , the calculator returns the value of 8.2558, indicating that our model predicts the ball will bounce 8.2557 centimeters on its 10^{th} bounce.

Similar keystrokes, but substituting 100, not 10, for *x*, will show us that on the 100^{th} bounce, our model predicts the ball will bounce 2.2284 times 10 to the negative seventh power. This, of course, is extremely small, but is not 0. (Another method on the calculator is described later.)

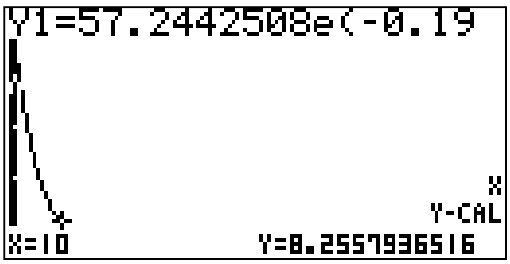
Students may find that we still don't obtain 0 even on the 1000^{th} bounce, at least according to our model. However, the calculator does return a value of 0 on the 10000^{th} bounce. Students may wish to find the bounce number upon which the calculator first predicts a height of 0. This exercise may provide a good context for discussing the fact that theoretically the ball will not stop bouncing because taking 83% of something, no matter how small, cannot be 0. Nevertheless, the calculator, though extremely precise, must use rounded values and will eventually produce 0.

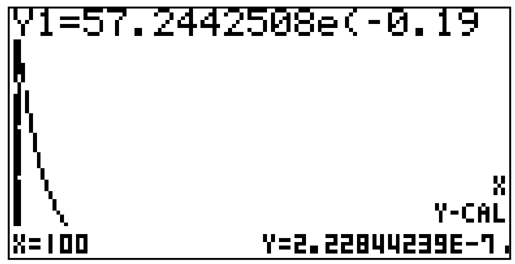
This idea, as well as finding $f(10)$ and $f(100)$, should also be explored graphically, as described below.

- Access the "Graph" menu. Our regression function should be displayed in Y1. If other functions appear, highlight them and either delete them by pressing [F2] and [F1] or de-select them by pressing [F1] . Functions that are de-selected have an equal sign that is not highlighted.
- Press [SHIFT] [F3] to access the Viewing Window. Set Xmin at 0, max at 110, and scale at 10. Leave the Y settings alone for now. Press [EXIT] and [F6] to draw the graph.
- With the graph in view, press [F2] to access the Zoom menu followed by [F5] so that the Y settings are adjusted automatically.

The calculator can find values quickly from the graph. To do so, with the graph in view:

- Press F5 to access the Graph Solver menu, F6 for more choices, and F1 for Y-CAL.
- Type in 10 for the *x* value and press EXE . You can watch the cursor move from left to right to find the desired point. Repeating this procedure, but entering 100 for *x* instead of 10, will find our other point. The two screens showing these values on our graph are shown below.





Watching the cursor move to the desired points should help students see clearly that, even though the model still predicts that the ball will bounce, the height of the bounce is so small that it is virtually negligible. The idea of a horizontal asymptote is appropriate here - the predicted height of the bounce becomes closer and closer to 0. Although the model may never predict 0 theoretically, for all practical purposes, the expected height is indeed 0.

D. Use the model to predict how many bounces it will take for the ball to reach heights of 5 and .01 cm. Show at least two methods of finding the solutions.

9) Find the appropriate x-values both numerically and graphically. If desired, also find them algebraically.

The first technique described here could have been used to address question 8) above as well. After the scatterplot and regression graphs have been drawn from the STAT menu:

- Call up the "Run" menu. Press [OPTN] and [F5] for STAT.
- Type in 5 (the *y*-value), [F1] (which asks the calculator to return the expected value for *x*), and [EXE] .

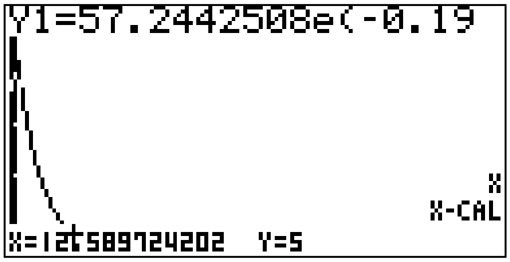
The calculator returns 12.5897, indicating that the 12th bounce should reach more than 5 centimeters, but the 13th bounce will be less than 5 centimeters. Similar keystrokes (.01 [F1] [EXE]) will tell us that the ball's maximum height will be .01 between the 44th and 45th bounces.

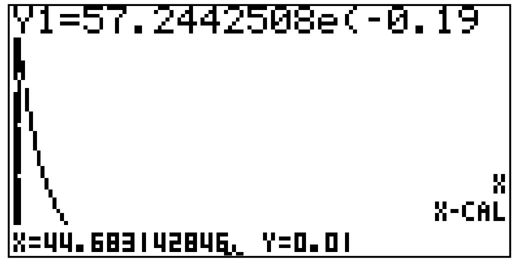
Note that for question 8), you could type in 10 for the 10th bounce, press [F2] to get the expected *y*-value, and [EXE] to see that the ball is expected to reach a maximum height of 8.2558 centimeters on the 10th bounce.

A graphic solution can also be obtained easily. From the GRAPH mode with the graph displayed (and the AUTO window set with *x*-values from 0 to 100):

- Press [F5] to obtain the Graph-Solver menu, [F6] to obtain more choices, and [F2] for an X calculation.
- Type in 5 after the "Y=" prompt and press [EXE] . You can watch the cursor move to the desired point.

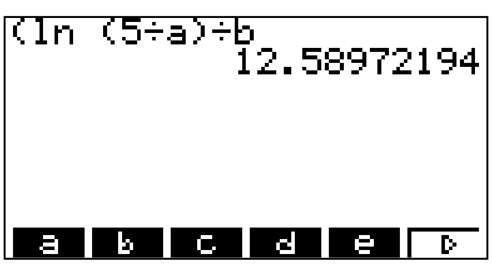
Similar keystrokes are used to find out which bounce produces a maximum height of .01 centimeters. The graphs displaying these points are shown below.

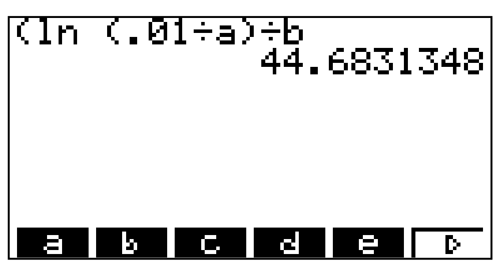




Algebraically, the solutions to our problems require the use of logarithms. The solution to $5 = 57.24 * e \wedge (-0.1936x)$ (when $y = 5$) is shown below.

- To isolate *x*, first divide both sides by 57.24. To avoid rounding errors, we will use the value for "*a*" which is stored in the calculator when the regression is performed. To have *a* and *b* available on the screen, from the "Run" menu, press [VARS], [F3] for STAT, and [F3] for GRPH. Now [F1] should produce "*a*" and [F2] should produce "*b*."
- We now have $\frac{5}{a} = e \wedge (bx)$, which we are trying to solve for *x*. To undo the *e*, we have to do the inverse function, which is the natural log function. Taking the natural log of both sides, we have $\ln\left(\frac{5}{a}\right) = bx$.
- All that is left to do is divide both sides by b, giving us $\frac{\ln\left(\frac{5}{a}\right)}{b} = x$. Using [F1] for "*a*" and [F2] for "*b*," our keystrokes are [ln] (5÷[F1]) ÷[F2] [EXE]. The screen is shown below left.
- To obtain the answer when $y = .01$, press the right or left arrow keys, and using INS, accessed with [SHIFT] [DEL], and [DEL], change the 5 to .01. The result is shown below right.





ENHANCING DATA COLLECTION WITH A QV-780

You may wish to use a digital camera for data collection. To do so, find an accessible wall that has a few clear feet in front of it. Attach a measuring tape vertically to the wall, and, if necessary, make large visible marks every 10 centimeters. You will make a movie of the bouncing ball, so find a position that allows you to see about 50 centimeters from the floor, the height from which you might choose to drop the ball. In order to minimize distortion from the angle at which the tape measure will be read, the person taking the movie should be about halfway between the floor and the height from which the object will be dropped. Practice bouncing the ball without spin so that it will have five or six "clean" bounces. To the extent that is possible, use a ball that is colorful (to make it easier to read the tape measure) and one that bounces back at least 70% of its original height.

Be consistent with the part of the ball you look for as you try to determine the maximum height of each bounce. Turn your camera on in RECord mode. Press [MODE] a sufficient number of times until you see the movie icon. Working with your partner, who will drop the ball from approximately 50 centimeters (or any other height that works well), make a movie that captures at least five bounces. This may take a few attempts.

Switch your camera to [PLAY] mode, and find the movie. Press the shutter at the start of the movie, then [MENU] to pause the action. Use the + key to advance the movie one frame at a time. Turn your calculator on and go to "List." In List 1, enter the bounce number, beginning with 0 (since it has not bounced when first dropped). In List 2, enter the maximum height of the bounce. The value for bounce 0 is the height from which the object was dropped. Construct the lists by advancing through the movie one frame at a time. You may wish to discuss with the students the effects of a digital image; a picture may not be taken when the ball is at its maximum height on any given bounce, so even with perfect technique, some errors in the data are inevitable.

Again, you may wish to avoid using technology for data collection. The problem can be fully explored by using the techniques described earlier. This will give students experience in estimation and interpolation and allows for even greater exploration into the errors of the model.

PROBLEM 2: SOUND AND THE DECIBEL SCALE

The intensity of sound is sometimes measured in watts per square meter. Humans can hear sounds that vary from approximately 10^{-12} to 100 watts per square meter. This range is extremely large; consequently we often use a different measure to assess the intensity of sound, the decibel. The decibel is one-tenth of a bel, a unit that was named after Alexander Graham Bell. Barely audible sounds are about 0 dB, city traffic or noisy kitchen sounds are around 90 or 100 dB, and a nearby jet may be about 140 dB.

The formula that relates the intensity of sound measured in watts per square meter (N) with the intensity measured in decibels (D) is $D = 10\log\left(\frac{N}{10^{-12}}\right)$.

A. Find the decibel equivalent for a conversation measured at $3.4 * 10^{-6}$ watts per square meter.

B. Find the intensity of sound measured in watts per square meter for a rock band whose music is measured at 128 dB.

ONE SOLUTION TO PROBLEM 2: SOUND AND THE DECIBEL SCALE

This problem can be investigated many ways. Here a graphic solution is suggested. We will let *x* represent the intensity of sound measured in watts per square meter and *y* be the intensity of sound measured in decibels. Our equation becomes

$$y = 10\log\left(\frac{x}{10^{-12}}\right) \quad .$$

From the MAIN MENU,

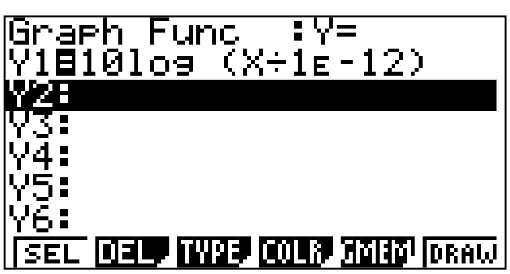
x Select “Graph.”

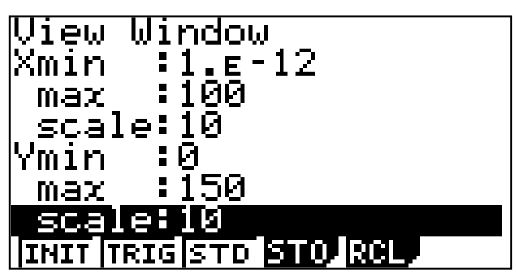
x Either delete any functions that are there (by highlighting them, pressing [F2] for delete, and [F1] to confirm), or de-select them by highlighting them and pressing [F1] .

x With the cursor at Y1, make sure the type is “Y=.” If not, press [F3] and [F1] .

x Type in the function so that it looks like the screen below left. Students should recall that 1 EXP –12 is the calculator’s scientific notation form for $1*10^{-12}$. Alternately, we could simply use the carat key for the exponent and type in 10^{-12} without using scientific notation.

As usual, the next step is setting up a reasonable window. We could start by using the values presented in the problem. Press [SHIFT] [F3] for the viewing window screen. Type in the minimum and maximum *x* and *y* values as suggested in the problem. At this point, the scales are not critical, but we have chosen 10 for both axes. Your screen may look as shown below right.



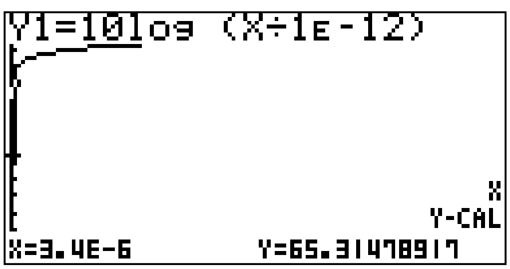


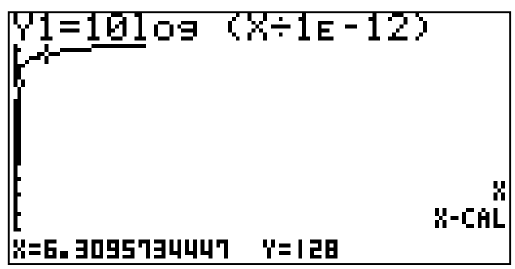
- From the viewing window, press EXIT and F6 to see the graph.

The problem asks us to find *y* when *x* is $3.4*10^{-6}$ and then to find *x* when *y* is 128. We can do this very easily with the calculator. For part A, while looking at the graph:

- Press F5 to access the graph solver.
- Press F6 for more and F1 for *y*-calculate.
- At the X= prompt, type in 3.4 EXP –6. Press EXE and you will see the solution, which tells us that this conversation would be approximately 65 dB. See below left.

For part B, the commands are similar; however, on the second step, press F2 for *x*-calculate instead of F1 . Type in 128 and press EXE , and you will see that the solution is approximately 6.3 watts per square meter. See below right.





PROBLEM 3: INVESTING MONEY

Suppose that you are 18 years old and have $1000 to invest. Explore the following numerically, graphically, and algebraically.

A. Determine the value of the investment when you are 60 if you invest the money at 3%, 6%, 10%, and 15%, assuming interest that is compounded annually.

B. Determine the value of the investment when you are 60 if you invest the money at 3%, 6%, 10%, and 15%, assuming interest that is compounded quarterly.

C. Determine the value of the investment when you are 60 if you invest the money at 3%, 6%, 10%, and 15%, assuming interest that is compounded monthly.

EXTENSIONS

1. What does continuous compounding mean? Determine the value of the investment if interest is compounded continuously.
2. Answer the questions above if you add $1000 to your investment every year.
3. Ask people and investigate the reasons for investing at different rates? Why do people invest money at 3% when higher interest rates are available? Are there compromises that must be made?

PROBLEM 4: EARTHQUAKES AND THE RICHTER SCALE

Two of the most significant earthquakes in the Western Hemisphere in recent years were the 1989 earthquake in the San Francisco area, the one which disrupted the World Series among other things, and the 1985 earthquake in Mexico City, which left thousands of people homeless. The Richter Scale is a logarithmic scale used to measure the magnitude of earthquakes. The Richter Number, R, is defined as $R = \log\left(\frac{I}{I_O}\right)$, where I_O is a standard value used for comparison purposes and I is the intensity of the quake. The two earthquakes mentioned above measured 7.1 and 8.1, respectively, on the Richter Scale. Compare their intensities.

"BUSTING BARRIERS" WITH THE ALGEBRA FX 2.0

Earlier, we needed to solve the equation $5 = 57.24 * e \wedge (-0.1936x)$. The ALGEBRA FX2.0 has the ability to solve such equations.

- From the MAIN MENU, choose "CAS."
- Press [F1] and the appropriate number for "Solve."
- Type in the equation as shown. When you use the [e^x] key, type in the exponent without the carat.
- Press [EXE] . The calculator returns the exact value of our solution.
- If we want an approximation of our answer, capture the right side of the solution and press the keys to "approximate answer." Press [EXE] .

Learning to Solve Equations

The ALGEBRA FX2.0 can also help students learn to solve equations. From the MAIN MENU, choose "CAS."

- To make sure all previous equations have been cleared, press the function for "Clear" and the appropriate number for "All Equations." Press [EXE] .
- Now type in the equation at the flashing cursor. When you press [EXE] , the equation will be entered as equation 1.
- Students can determine for themselves how to solve the equation. To divide both sides of eqn(1) by 57.24, simply enter the division sign and 57.24. Then press [EXE] . The result becomes eqn(2).
- To simplify this, press [F1] and the appropriate number for "Simplify." Press [EXE] and the ALGEBRA FX2.0 simplifies both sides of your equation.

Continue in this manner until the equation has been solved. Even if you tell the calculator to do something that does not lead to a solution, the calculator will do it; you should just notice that it has not helped you get where you want to go!

TEXT SECTION CORRESPONDENCES

The materials in this module are compatible with the following sections in the listed texts.

TEXT	SECTION
AWSM – Focus on Algebra (1998)	10.2
AWSM – Focus on Advanced Algebra (1998)	9.1, 9.2
Glencoe – Algebra 1 (1998)	11.4, 11.5
Glencoe – Algebra 2 (1998)	10.1, 10.1B, 10.2, 10.6, 10.7
Holt Rinehart Winston – Algebra (1997)	10.5, 10.6
Holt Rinehart Winston – Advanced Algebra (1997)	7.1, 7.2, 7.3, 7.4, 7.7
Key Curriculum – Advanced Algebra Through Data Exploration	7.1, 7.3, 7.8
Merrill – Algebra 1 (1995)	4.3
Merrill – Algebra 2 (1995)	12.1, 12.2, 12.7, 12.8
McDougal Littell – Algebra 1: Explorations and Applications (1998)	9.2, 9.3
McDougal Littell – Heath Algebra 1: An Integrated Approach (1998)	8.2, 8.6, 8.7, 12.3
McDougal Littell – Algebra: Structure and Method Book 1 (2000)	
Prentice Hall – Algebra (1998)	8.1, 8.2, 8.3, 8.4
Prentice Hall – Advanced Algebra (1998)	7.1, 7.2, 7.3, 7.5
SFAW: UCSMP – Algebra Part 1 (1998)	
SFAW: UCSMP – Algebra Part 2 (1998)	8.2, 8.3, 8.4
SFAW: UCSMP – Advanced Algebra Part 1 (1998)	
SFAW: UCSMP – Advanced Algebra Part 2 (1998)	7.3, 9.1, 9.2, 9.3, 9.4, 9.5, 9.6, 9.7
Southwestern – Algebra 1: An Integrated Approach (1997)	