CLEMSON ALGEBRA PROJECT
UNIT 16: SEQUENCES

PROBLEM 1: GOING INTO DEBT

In order to purchase a used car, you need to borrow $5,000. You decide to pay with a credit card, which, as many credit cards do, charges you 1.5% per month (18% per year) on the unpaid balance. You decide that you can afford to pay the credit card company $100 per month.

A. How much do you owe after 1 year? How much money have you paid?

B. How much do you owe after 3 years? How much money have you paid?

C. Answer questions A and B assuming you pay $200 per month.

D. Answer questions A and B assuming you are charged only 0.75% per month in interest.

E. Explore similar questions and reflect upon what you have found. Write a short paragraph discussing these results.

MATERIALS

Casio CFX-9850Ga PLUS or ALGEBRA FX2.0 Graphing Calculator

EXTENSIONS

Find out the interest rates charged by different credit card companies, the bank, or other institutions you might use to borrow money. Explore the effects these rates have on the total amount of money you have to pay back.

ONE SOLUTION TO PROBLEM 1: GOING INTO DEBT

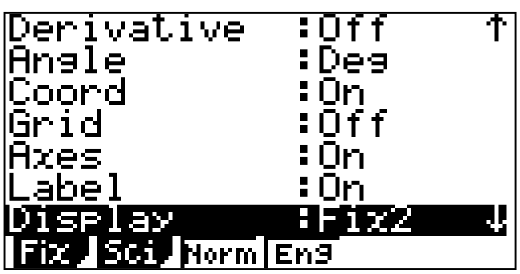
A. How much do you owe after 1 year? How much money have you paid?

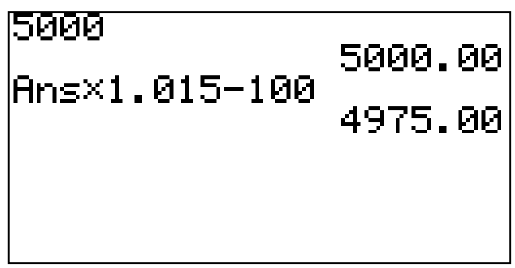
One way to begin this problem is by using the “RUN” mode on the calculator. From the MAIN MENU, choose “Run.” Then,

- Press [SHIFT] [MENU] for the SET UP.
- Scroll down to Display.
- Press [F1] for Fix, and press [F3] for two decimal places. Since we are dealing with money, this will automatically round our results to the nearest cent. See the screen below left.

Now we are ready to work on the problem. Press [EXIT] to return to the home screen. Then:

- “Seed” the calculator by typing in 5000 and pressing [EXE] .
- Next month we will owe 101.5% of this value minus the $100 we will pay. Using the automatic answer generated by the calculator, all we need do is multiply by 1.015 and subtract 100. See the screen below right.





This tells us that after one month, although we have paid $100 toward our $5,000 bill, we still owe $4,975. Pressing [EXE] 11 more times tells us that though we have paid $1,200 over the course of the year, we still owe $4,673.97. In other words, our bill is only $326.03 less than what it was when we began the year!

B. How much do you owe after 3 years? How much money have you paid?

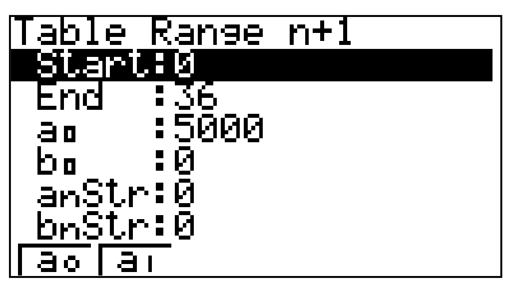
We could solve Part B using the same technique we used for Part A. However, this problem, one in which we use a result to generate a new result (a process called recursion), can be explored more easily with a different technique. To begin, from the MAIN MENU, choose “Recur.”

The problem we are working on is called a first order problem; in other words, we only need one previous result to determine the next value. From the primary “Recursion” screen, press SHIFT MENU to access the SET UP. Make sure that the first option, Σ Display, is Off. Then,

- Press F3 for TYPE.
- Press F2 for a_{n+1}, which tells us that we can generate the (n + 1)st value by knowing the nth value.
- Now press F4 to access previous terms, F2 for a_n, and then multiply this by 1.015 and subtract 100. Press EXE . The result is shown below left.

We now wish to explore this sequence. One way of doing so is with a table. To set up the range of the table:

- Press F5 from the screen shown below left.
- We want to start with a_0, since at the beginning no months have gone by. Consequently, if we are looking for the first 36 months (3 years), we can set the Start value at 0 and the End value at 36. We also need a_0, the value of our loan, to be 5000. (Because we are working with only one recursion problem, we can ignore the b values. We are also not looking at convergence or divergence, so we can also ignore the anStr and bnStr values.) The table range values are shown below right.



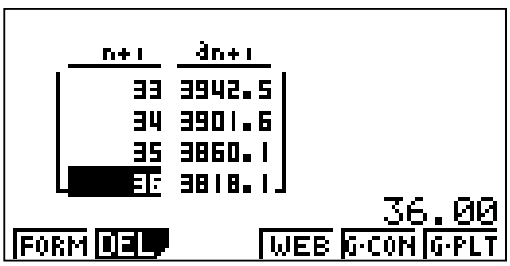
x Press [EXIT] and [F6] to go to the table.

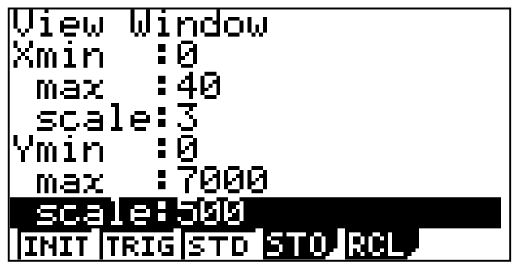
Using the down arrow key to scroll down through the table, you can determine that, although you have paid a total of $3,600 for the $5,000 you borrowed, you still owe $3,818.10 (see below left). In other words, you have only paid back $1,181.90 of the principal!

Another way to look at this is with a graph. Our horizontal axis represents n, the number of payments you have made. The vertical axis, a_n, represents the amount you still owe on the loan.

Before we look at the graph, we need to set our window. From the table,

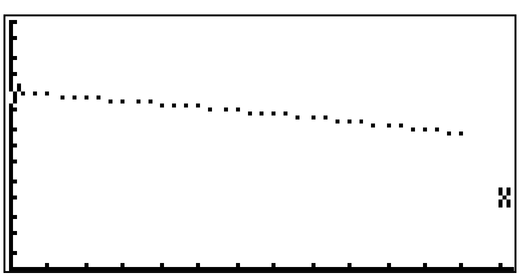
x Press [SHIFT] [F3] to access the View Window. Put in appropriate values (some possible values are shown below right). When you finish entering the values, press [EXIT] to return to the primary “Recursion” screen.

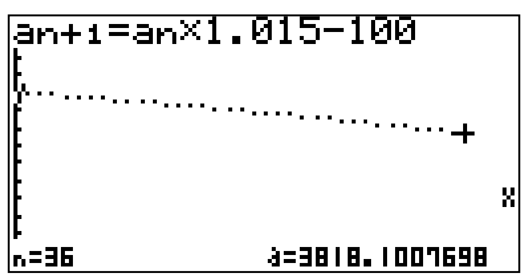




x Press [F6] to reenter the table and [F6] for a graph plot. See below left.

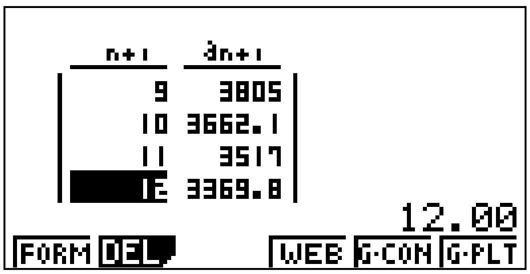
The TRACE feature, accessed by pressing [F1] from the graph, can be used to move from one value to another. Below right shows the result when the cursor is moved to the value showing the amount owed after 36 months.

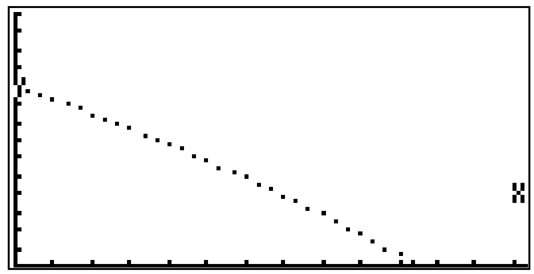




C. Answer questions A and B assuming you pay $200 per month.

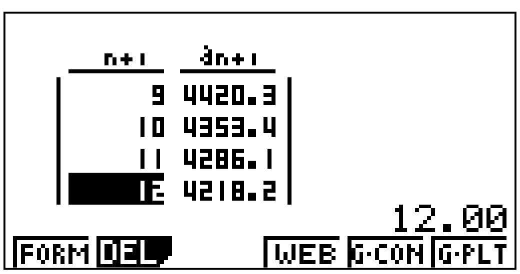
To investigate this problem, we need to change the formula slightly. Press EXIT twice to return to the formula. Instead of subtracting $100, now subtract $200. The table values or viewing window need not be changed. The results tell us that after 12 months of paying $200 per month, we would still owe $3,369.80 (having now paid off $1,630.20 of the principal). After 36 months we would owe -$909.40; in other words, we have more than paid back the loan. The table values around 12 months and the graph are shown below.

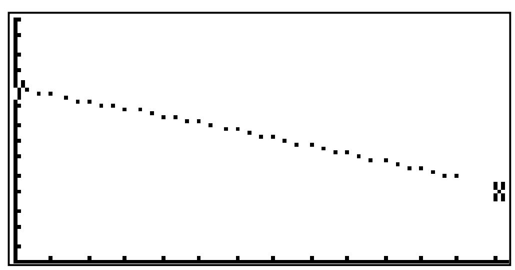




D. Answer questions A and B assuming you pay only 0.75% per month in interest.

To explore the change the interest rate has on the original problem, multiply a_n by 1.0075 and subtract $100. At 12 months we owe $4,218.20 and at 36 months $2,427.90. Note that after paying $3,600, we've done much better with this interest rate, having paid off $2,572.10 of the original loan as compared to the $1,181.90 we had paid when charged at the 1.5% rate.





E. Explore similar questions and reflect upon what you have found. Write a short paragraph discussing these results.

Results will vary. However, students should note how much difference paying more each month or getting a lower interest rate can make in terms of the balance that remains or in terms of how much is needed to pay off the entire debt.

One point of discussion that may be of interest is the effect that paying with a credit card may have upon the cost of the car. Because major credit card companies charge the sellers a fee for any transaction, car dealers who accept credit cards may not be as willing as others to negotiate on the price of the car.

PROBLEM 2: THE FIBONACCI SEQUENCE

The Fibonacci Sequence is one that connects mathematics to nature and to art. Students may be interested in researching this and making a class presentation on it. The sequence itself is a second order sequence; in other words, it uses two previous values to generate the next value. To begin, we need two initial values, which we now identify as 1 and 1. To generate the next value in the sequence, simply add the two previous values. The sequence thus can be represented as {1, 1, 2, 3, 5, 8, …}.

A. Generate the first 25 values for the sequence and graph them.

B. Find the ratio of the 25^{th} term to the 24^{th} term, and compare this to earlier ratios in the sequence.

C. Instead of seeding the sequence with initial values both 1, try different seeds. Then generate the first 25 values for the sequence and graph them.

D. Again find the ratio of the 25^{th} term to the 24^{th} term, and compare this with the ratio you found in part B.

E. The Golden Ratio, the ratio often considered most pleasing to the eye and one that has been used in a great deal of art, is $\frac{1+\sqrt{5}}{2}$. Compare your results from parts B and D to this value.

EXTENSIONS

A wealth of information is available on the Fibonacci Sequence, the Golden Ratio, and the Golden Rectangle. Students interested in art and/or nature may wish to explore these subjects in greater detail and prepare a presentation for the class. In addition to traditional references, the Internet has many web sites with an abundance of information and pictures on this topic.

ONE SOLUTION TO PROBLEM 2: THE FIBONACCI SEQUENCE

A. Generate the first 25 values for the sequence and graph them.

From the MAIN MENU, choose “Recursion.” Then,

x Press F3 for TYPE.

x Press F3 for second order recursion.

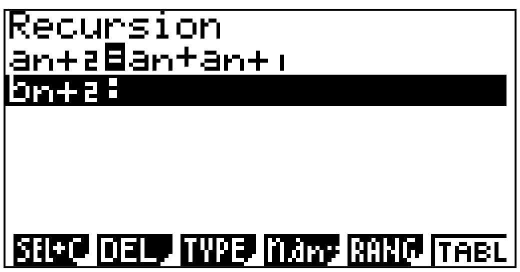
x Before entering the formula, press F4 to be able to access previous terms in the sequence.

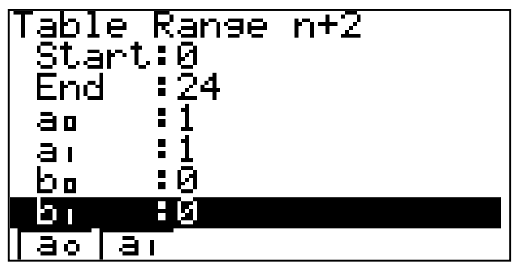
x Put in the formula $a_{n+2}=a_n + a_{n+1}$, using the appropriate function keys. Press EXE . The result is shown below left.

To set the range,

x Press F5 for RANGE.

x Set the Start at 0, the End value at 24 (the 1st term is a_0), and both a_0 and a_1 at 1. The values are shown below right. Press EXIT when finished.



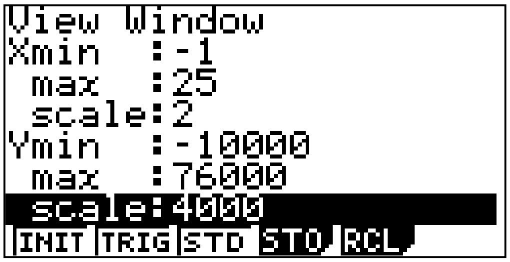


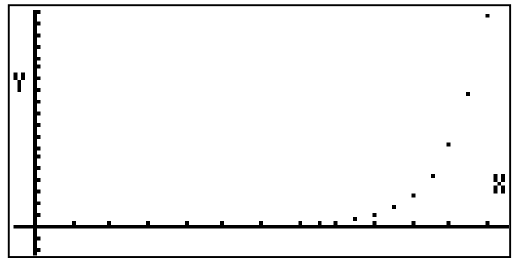
After performing the commands above, you should be back at the primary “Recursion” screen. Pressing F6 will show the table. Scrolling down through the table reveals that a_{24}, the 25th term of the sequence, is 75,025. Before graphing the sequence, we should use this information to set the window.

x Press SHIFT F3 to access the window.

x One set of possible values is shown below left. Press EXIT when finished.

x Press F6 to reenter the table and then F5 to see a graph with the points connected, selecting the a_n option. The graph is shown below right. Notice how steep the graph becomes.





From the graph, pressing [F1] followed by the right arrow key allows you to TRACE through the values.

B. Find the ratio of the 25th term to the 24th term, and compare this to earlier ratios in the sequence.

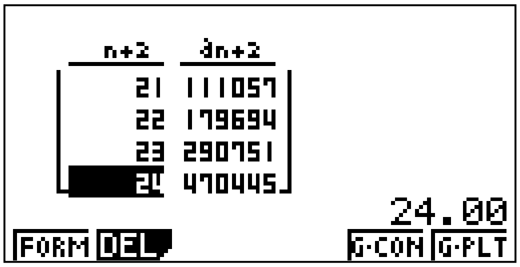
The table shows the first 25 numbers in the sequence: {1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46358, 75025}. Although initially students have no indication of why we might be interested in studying the ratio of successive terms, the connections will be made later for them. The ratios of a_{n+1}:a_n for these terms, rounded to three decimal places, are listed below. (The values were obtained using the “Run” menu.)

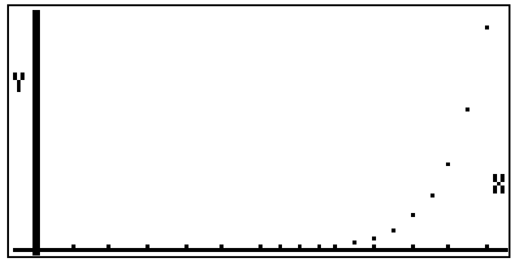
{1, 2, 1.5, 1.667, 1.600, 1.625, 1.615, 1.619, 1.618, 1.618, 1.618, 1. 618, …}

Clearly, the ratios are converging at 1.618.

C. Instead of seeding the sequence with initial values both 1, try different seeds. Then generate the first 25 values for the sequence and graph them.

Students will likely choose different seed values. To set different seeds, from the primary “Recursion” screen, call up the Table Range and change the values for a_0 and a_1. For the example here, a_0 and a_1 were set at –3 and 12, respectively. The last few values are quite different from the original (see below left), but the graph looks very much like the original (see below right). The window for the graph was changed so that the maximum y-value is 500,000.



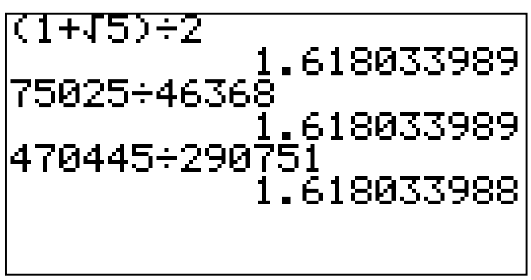


D. Again find the ratio of the 25th term to the 24th term, and compare this with the ratio you found in part B.

The ratio of the 25th to 24th terms, 470445:290751 is, once again, 1.618. Despite the different seeds and the different values for the terms, the end ratio remains the same. Students may wish to explore how quickly the ratio of the terms converges towards 1.618.

E. The Golden Ratio, the ratio often considered most pleasing to the eye and one that has been used in a great deal of art, is $\frac{1+\sqrt{5}}{2}$. Compare your results from parts B and D to this value.

From the "Run" menu, if necessary change the SET UP so that the display is normal. The values of $\frac{1+\sqrt{5}}{2}$, 75025:46368 (the ratio of the 25th to 24th terms with seeds of 1 and 1), and 470445:290751 (the ratio of the 25th to 24th terms with seeds of –3 and 12), are shown on the screen below. Note that the values are identical to 9 significant digits. The ratio of consecutive terms of the Fibonacci Sequence converges to $\frac{1+\sqrt{5}}{2}$. This value is called the Golden Ratio.



PROBLEM 3: GETTING RICH

Here's a classic problem. Suppose you were to work every day for a stretch of 30 days. You have two pay options: you can be paid $1,000 for every day that you work, or you can be paid $0.01 the first day, $0.02 the second day, $0.04 the third day, $0.08 the fourth day, and so on. Which pay plan would you choose and why?

HINT 1: From the main "Recursion" screen, press [SHIFT] [MENU] to enter the SET UP and turn the first option, Σ Display, On.

HINT 2: Use a_n for one sequence and b_n for the other.

PROBLEM 4: TAKING YOUR MEDICINE

Suppose that you have an illness that requires you to take 200 milligrams of a medication every 12 hours. Further suppose that during a 12-hour period your body disposes of 80% of the medication in your bloodstream. How much medication is in your bloodstream after 24 hours? After 48 hours? In the long run?

Explore the underlying ideas by changing the dosage and the percent of medication disposed of by your bloodstream.

EXTENSION

Suppose you forget to take your medication one day. Explore what happens if

A. you just skip that day, or

B. you try to make up for the day by taking two doses on the following day.

ENHANCING YOUR WORK WITH A DIGITAL CAMERA

In addition to having students show pictures of themselves solving the problems, a digital camera can be used to enhance the solutions. For ***GOING INTO DEBT***, students might take pictures of credit card statements showing different rates of interest. They should, of course, be discreet about the information their pictures reveal, avoiding credit card numbers and the amount of individuals' debts.

A more exciting use of the camera involves work on ***THE FIBONACCI SEQUENCE***. After students have conducted their research, they can take pictures of examples they find in nature. For instance, they might take pictures of the spirals on a daisy, using the Macro setting (the flower icon) on the camera for a close-up. As they should learn, examples abound in nature, and the documentation they can provide with the digital camera can help students see how mathematics is truly connected to the world around them.

TEXT SECTION CORRESPONDENCES

The materials in this module are compatible with the following sections in the listed texts.

TEXT	SECTION
AWSM – Focus on Algebra (1998)	10.1
AWSM – Focus on Advanced Algebra (1998)	4.1, 4.2
Glencoe – Algebra 1 (1998)	
Glencoe – Algebra 2 (1998)	8.7, 11.1, 11.2, 11.3, 11.4, 11.5, 11.6
Holt Rinehart Winston – Algebra (1997)	1.1, 10.6
Holt Rinehart Winston – Advanced Algebra (1997)	12.1, 12.2.,12.3
Key Curriculum – Advanced Algebra Through Data Exploration	1.1, 1.2, 1.3, 1.4, 1.5, 5.2
Merrill – Algebra 1 (1995)	
Merrill – Algebra 2 (1995)	13.2, 13.4
McDougal Littell – Algebra 1: Explorations and Applications (1998)	
McDougal Littell – Heath Algebra 1: An Integrated Approach (1998)	
McDougal Littell – Algebra: Structure and Method Book 1 (2000)	
Prentice Hall – Algebra (1998)	
Prentice Hall – Advanced Algebra (1998)	12.1, 12.2, 12.3
SFAW: UCSMP – Algebra Part 1 (1998)	
SFAW: UCSMP – Algebra Part 2 (1998)	8.3
SFAW: UCSMP – Advanced Algebra Part 1 (1998)	1.8, 1.9, 3.7
SFAW: UCSMP – Advanced Algebra Part 2 (1998)	7.5
Southwestern – Algebra 1: An Integrated Approach (1997)	