Unit 2

Rational Functions, Limits, and Asymptotic Behavior

Introduction

An intuitive approach to the concept of a limit is often considered appropriate for students at the precalculus level. In this unit, discovery is used as a method to help students become comfortable with the notion of limits, and the term *approaches* is used instead of the word *limit*.[1]

Problem 1

Each year the Josten's Company publishes the Walhalla High School yearbook, the *Walhira.* The company charges an $8000 fixed cost to set up their printers according to our school's specifications. Additionally, we are charged $60 per copy (unit cost) to print the yearbooks.

1. Write a *function* that would describe the cost per copy of the yearbook.
2. A *rational function* is defined as a function *f* which can be written in the form *f*(x) = $\frac{p(x)}{q(x)}$ where p and q are polynomial functions and *q(x)* ? 0. Explain why your cost function is or is not a rational function.
3. There are 850 students at Walhalla High School and 60 members of the faculty and staff. What would constitute an appropriate *domain* for x? What would be an appropriate *range*? Explain your choices.
4. Does the function have a *x-intercept*? If so, use the equation solver capability of your calculator to find it. Explain its meaning in terms of yearbooks. Does the function have a *y-intercept*? Explain.
5. Is the cost function *even, odd or neither*? Explain what your conclusion means in terms of the graph of your function.
6. Fractional values of x have no meaning for our function. However, it is interesting to observe the behavior of our function as x approaches 0. Use the table capability of

your calculator to evaluate the value of our function for values of the independent variable that are close to 0 (.1, .01, .001, .0001, .00001, etc.) What happens to the function as x approaches 0?

?? The line x = a is a *vertical asymptote* of a function f if f(x) ? ? ? as x ? a from the right or from the left.

?? Use this definition and your table values to determine any vertical asymptote for our function.

7. Considering the student population at Walhalla High School, extremely large values of x are unreasonable. However, the behavior of our function is interesting as x ? ? . Use the table to evaluate large values of x (1000, 10000, 100000, 1000000, 10000000). What value does our function appear to approach as x gets larger? Explain in terms of our yearbooks.

?? A *horizontal asymptote* (f(x) = c) occurs in a rational function when f(x) ? c as x ? ? ? .

?? Use this definition and your table values to determine any horizontal asymptotes for our function.

8. Confirm your findings in 1 – 7 by graphing your function using the domain and range you have chosen in your window.

One Solution

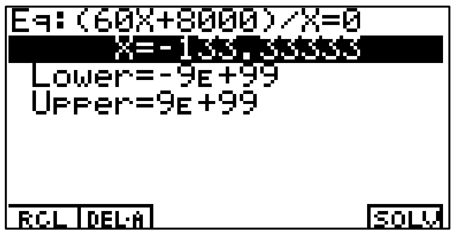
1. c(x) = $\frac{60x ? 8000}{x}$

2. c(x) is a rational function because it can be written as the ratio of p(x) = 60x + 8000 (the total cost of the yearbooks) to q(x) = x (the number of yearbooks ordered).

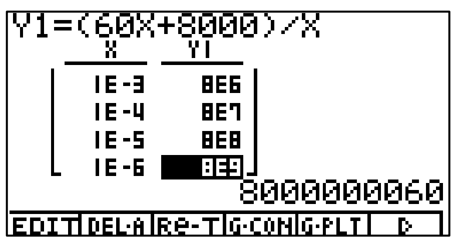
3. Answers may vary. Answers with reasonable justifications are acceptable. The domain might include numbers greater than zero and less than or equal to 920. It is possible that all the students as well as all the members of the faculty and staff would purchase a yearbook. There are also extra books ordered every year for the library and members of the community. The range might include numbers from 60 to 8060.

If only 1 book were ordered, the cost would be $8060; if 920 books were ordered, the cost would be around $68.

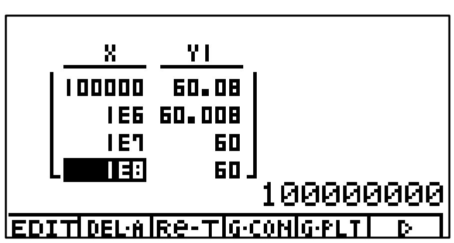
4. The x-intercept is –133.333333… If the school purchased –133.33… yearbooks the cost would be zero. This value is meaningless in terms of yearbooks. If x = 0 our function is undefined. Therefore there is no y-intercept.



5. Choose CAS on the menu and use substitute to show that c(-x) ? c(x). The function is not even and would not be symmetric with respect to the y-axis. c(-x) ? -c(x), ? the function is not odd and would not be symmetric with respect to the origin. It is neither.
6. x = 0 is a vertical asymptote of c because c(x) ? ? as x ? 0. Fractional numbers of yearbooks has no meaning.



7. c(x) ? 60 as x ? ? . As the number of yearbooks gets larger, the $8000 set-up fee charged by the company has less and less effect on the unit price of the yearbooks. The marginal rate of the cost of the yearbook would approach the cost of the paper.



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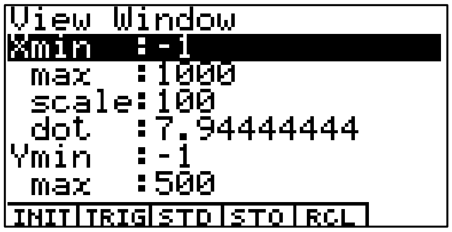
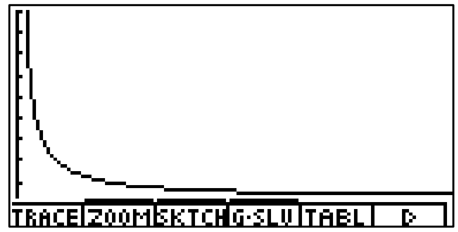




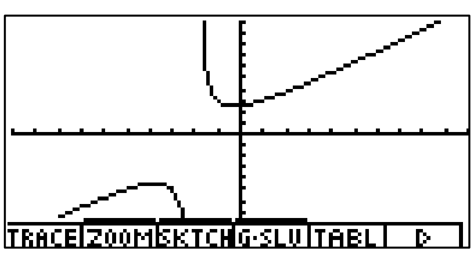
Problem 2[2]

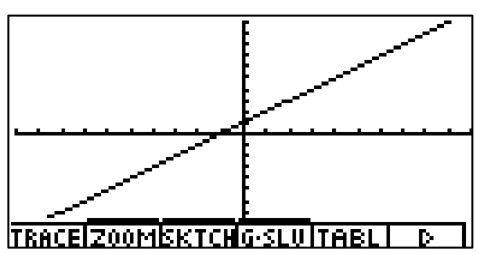
$f(x) = \dfrac{(x^2\ ?\ 3x\ ?\ 5)}{(x\ ?\ 2)}$ $\qquad$ $g(x) = \dfrac{(x^2\ ?\ 3x\ ?\ 2)}{(x\ ?\ 2)}$

1. Compare and contrast *f* and *g* with respect to domain, range, intercepts, and any asymptotes.
2. (True or False) Every rational function has a vertical asymptote. Explain.

One Solution

1. At first inspection, the equations of the functions *f* and *g* appear to be quite similar. They have exactly the same denominator and their numerators are identical except for the constant term. However, if we graph the functions the distinctions become evident. The first one looks like a typical rational function with a vertical asymptote and a slant asymptote. The graph of the second function, g, is a straight line. In *g* the denominator is a factor of the numerator so when x ? -2, g(x) = x + 1.





?? Rational functions are not defined for the zeros of the denominator, therefore the domains of the functions are the same ?x|x ? ?2?.

?? If we look at the values of *f* when x is close to –2, we see that *f* ? ?? as x ? -2 from the left and from the right. *f* has a vertical asymptote at x = -2. On the other hand, g is close to –1 as x ? -2 from the left and right. *g* has a *hole* in the graph at $x ? ?2$.

?? Using long division, we find that *f* can be written as $f(x) ? x ? 1 ? \frac{3}{x ? 2}$. Since $\frac{3}{x ? 2} ? 0$ as $x ? ?$ so *f* has a slant asymptote at $y ? x ? 1$.

?? *f* has no x-intercepts; the x-intercept of *g* is –1. The y-intercept of *f* is 2.5; the y-intercept of *g* is 1.

2. False.

?? If the degree of the numerator is exactly one more than the degree of the denominator and if the denominator is a factor of the numerator, there is a hole in the graph at the zero(s) of the denominator.

?? If the degree of the numerator is exactly one more than the degree of the denominator and if the denominator is not a factor of the numerator, the function has a slant asymptote at the zero(s) of the denominator.

Problem 3[3]

Slant Asymptotes

If the degree of the numerator of a rational function is exactly *one more* than the denominator, the graph of the function has a **slant asymptote** which means that there is a straight line that the graph of the function approaches as $x ? ??$. Look carefully at the two solutions to the following problem:

Find the slant asymptote of the curve with equation

$$f(x) = \frac{(x^2 ? 3x ? 7)}{(x ? 2)}$$

Solution 1: By division, we find that

$$f(x) = \frac{(x^2 \ ? \ 3x \ ? \ 7)}{(x \ ? \ 2)} = x + 1 + \frac{5}{(x \ ? \ 2)}$$

Since the last term approaches zero as x approaches infinity, the asymptote is the line $y \ ? \ x \ ? \ 1$.

Solution 2: Dividing the numerator and denominator by x, we find that

$$\frac{(x^2 \ ? \ 3x \ ? \ 7)}{(x \ ? \ 2)} = \frac{x \ ? \ 3 \ ? \ \frac{7}{x}}{1 \ ? \ \frac{2}{x}}$$

When x is large, the value is approximately x + 3, so the asymptote should be the line with equation y = x + 3.

1. Determine which (if either) solution is correct, and develop a convincing argument for your choice.
2. Develop a general theory to explain what is going on. Show how students could generate other examples of this phenomenon.

Problem 4

Since you have a new laptop with publishing capabilities, the yearbook staff offers you a part time job formatting the text on the pages of the *Walhira*. According to the agreement, you will be paid $500 for the job.

1. Write a function that will describe your hourly wage.
2. Explain why your function would be rational.
3. Generate a table which would report your hourly wage if it took 5, 10, 15,…100 hours to complete the job.
4. What domain and range would be appropriate for your function?
5. Is your function even, odd or neither? How would this affect the graph of your function?
6. What are the intercepts? Explain in terms of the job.
7. Find any horizontal or vertical asymptotes. Explain their meaning in terms of the job.
8. At what point would you be better off working at Hardee's for minimum wage?

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[1] Demana, F., Waits, B., Clemens, S., Foley, G. (1997), PRECALCULUS: A Graphing Approach. Addison-Wesley Publishing Company.
[2] TICAP (1995), TICAP: Technology Intensive Calculus For Advanced Placement. Clemson, SC: Clemson University.
[3] TICAP (1995), TICAP: Technology Intensive Calculus For Advanced Placement. Clemson, SC: Clemson University.