Unit 3
Exponential Functions

Introduction

Exponential models arise from data whose outputs form a geometric progression. Any change in a quantity that results from repeated multiplication generates an exponential model. When the repeated multiplier is greater than one, the change is called exponential growth. When the repeated multiplier is less than one, the diminishing amount is referred to as exponential decay.

Examples of exponential growth and decay occur often in the real world. In the world of finance, for example, there are savings accounts, mortgages, automobile loans. Population growth and the half-lives of radioactive material also provide numerous real world examples of exponential growth and decay. An investigation is included in this unit which will provide students with a good introduction to the general behavior of exponential functions.

Problem 1

Exploring the "Dynamics" of Exponential Functions
Let's consider exponential functions of the form:

$$f(x) = ab^x,\ b>0$$

1. Work through the following procedure to explore the effect of "a" on the exponential function above:
   - ?? Choose the dynamic feature on the menu.
   - ?? Enter the function above in y_1.
   - ?? Select "A" as the dynamic variable. Let B = 2.

?? V-Window:

-10 ? x ? 10

-25 ? y ? 25

y-scale 5

?? The range:

start (-10)

end (10)

pitch (4).

?? Select *Slow* speed.

?? Press DYNA.

?? Observe.

2. Describe the graphs.
3. What effect does the sign of "A" have on the graph?
4. Change the V-Window to –15 ? y ? 15 and repeat. What effect does "A" have on the value of the function? Keep B constant at 2 and choose other ranges and pitches for "A" until you are sure that you can articulate the effect of "A".
5. Vary the V-Window and the range to verify you conjectures.
6. Work through the following procedure to explore the effect of "b" on the exponential function above:

?? Select "B" as the dynamic variable. Let A = 2.

?? Select the standard V-Window.

?? The range:

start (0.1)

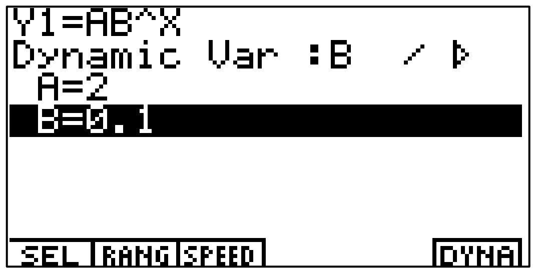
end (2.0)

pitch (0.2).

?? Select *Slow* speed.

?? Press DYNA.

?? Observe.



7. Describe the graphs when 0 < b < 1.
8. Describe the graphs when b > 1. What happens when b = 1?
9. Vary the V-Window and the range to verify you conjectures.

One Solution

1. Observation
2. As x gets larger, the graphs all intersect the y-axis at “A” and then increase or decrease. As x gets smaller, the graphs appear to approach the x – axis.
3. When “A” is negative, the graph decreases without bound. When “A” is positive the graph increases without bound.
4. “A” is the value of the function when x = 0. This is often considered the *initial value* of an investment, a loan, or a population for example.
5. Observation
6. Observation
7. If “B” is between 0 and 1, the function decreases. This behavior is known as exponential decay. Multiplying repeatedly by a fraction will result in a decrease.
8. If “B” is greater than 1, the function increases. This is exponential growth. Multiplying repeatedly by a value greater than 1 will result in an increase.
9. Observation

Problem 2

Suppose your mother had been far-sighted enough to predict the recent economic expansion and bull market on Wall Street and on your 6th birthday had invested part of your college money in the stock market. She was conservative with your money and chose dependable "blue chips." Each year on your birthday, you and she celebrate your earnings and record the value of your investment on your wall chart along with your height. The investment data are recorded in the following chart:

1990	1000
1991	1140
1992	1298.46
1993	1484.12
1994	1687.47
1995	1920.34
1996	2189.18
1997	2497.85
1998	2852.56
1999	3249.06
2000	3703.93

Input the data into the lists and draw a scatter-plot.

1. What is the general appearance of the data? Does it appear that your investment has achieved exponential growth? Does it bear resemblance to any of the models in the investigation? Explain.
2. Develop a model for your investment. What value would you choose for *a*? Explain your choice. "Guess" a function that would model your investment using the Graph

Function menu to input functions you want to overlay on the statistical graph. Continue until you have achieved a well fitting curve. Record your function.

3. What would constitute an appropriate domain and range for your function? Explain your choices.
4. Interpret the meaning of the intercepts (if they exist) in the context of our investment function.
5. Is our investment function even, odd or neither? Explain.
6. Describe the behavior of your investment function as x ? ? ? .
7. Suppose your mother had chosen to invest your money in a CD instead. At the local savings and loan they offered 6% compounded quarterly. The *compound interest formula* is an exponential function, of course. P dollars invested at r% compounded n times a year for t years would be:

$$A(t) = P(1+\frac{r}{n})^{nt}$$

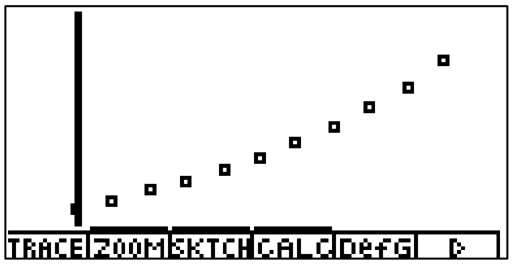
How would you have done at the savings and loan?

8. A local bank advertises that they pay 5.9% but compound interest monthly. Would you have done better there than at the savings and loan? Explain.
9. At the end of the year the Dow Jones Industrials closed at the values listed in the following table. How did you do compared to the Dow?

1990	2633
1991	3168
1992	3301
1993	3754
1994	3834
1995	5117
1996	6448
1997	7908
1998	9181

1999	11497
2000	??????

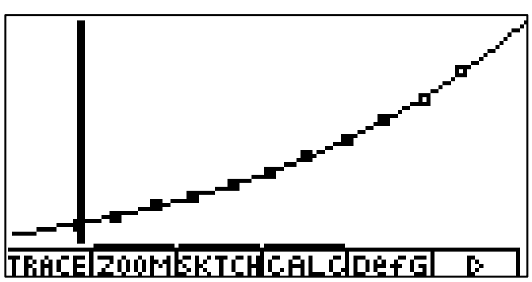
One Solution



1. The scatter plot appears to exhibit exponential growth. Looking more closely at the data and calculating the differences reveals an annual increase of roughly 14% which would indicate a repeated multiplier of 1.14.
2. Since the initial value of the investment is $1000, the value of *a* should be 1000. The investment is growing at an annual rate of 14% so *b* should be 1.14.

   $f(x) = 1000(1.14)^x$

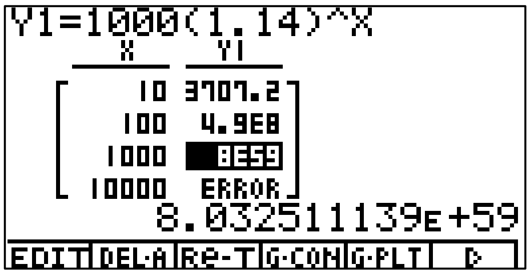
   It would be unusual, of course, to achieve such consistent growth in the stock market.

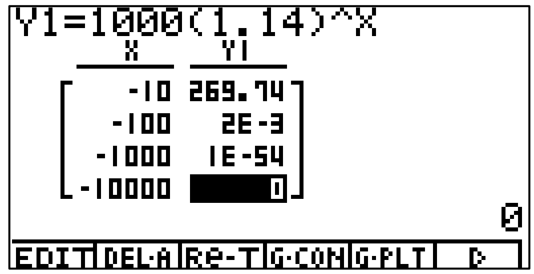


3. The domain of the function $f(x) = 1000(1.14)^x$ which models our situation is all reals. However, negative values of x have no meaning in the context of this situation. Even

so, one might ask such questions as “ What investment in 1989 would have yielded a return of $1000 in 1990 if the investment had grown at 14%.”

4. The y-intercept is 1000, the original investment. There is no x-intercept.
5. Choose CAS on the Menu and substitute (-x) for x. It is neither.
6. y? ? as x? ? and y ? 0 as x? - ?

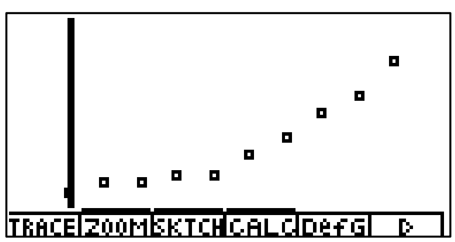


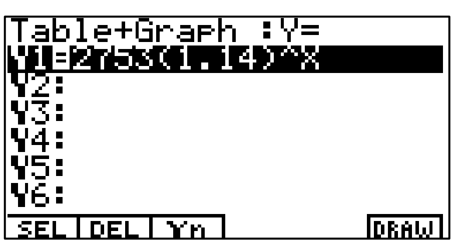


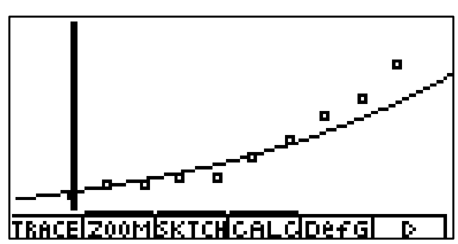
7. $A(10) = 1000(1+\frac{.06}{4})^{4(10)} = \$1814.02$
8. No, worse. The slight increase in number of times your interest was compounded was more than offset by the decrease in rate of interest.

   $A(10) = 1000(1+\frac{.059}{12})^{12(10)} = \$1801.38$

9. Actually the Dow did a little better.







Problem 3

Calculate the value of your $1000 investment in Problem # 2 at 6% if the interest was compounded:

Yearly	
Semiannually	
Quarterly	
Monthly	
Weekly	
Daily	
Hourly	
Every minute	
Every second	
Continuously	????????

1. Are you surprised that your earnings did not increase any more than they did? Consider an investment of $1 for 1 year at 100% interest. As the number of times the interest is compounded increases, how does the investment grow? Graph this function and trace its behavior as n ? ? .

$$y = 1(1 + \frac{1}{n})^{n}$$

2. What number does it appear to approach? The limit of this function as n ? ? is the irrational number *e* discovered by the Swiss mathematician Leonard Euler. The formula for *continuously compounding interest* is:

$$A(t) = Pe^{rt}$$

Compute the value of your investment if the interest were compounded continuously.

Problem 4

At the Ski Lodge

You and your friend are at a Ski Lodge in Boone, NC. After a couple of hours on the slopes, you take a “hot chocolate break.” If the temperature outside is 30? and the temperature inside is 72?, how much longer will the hot chocolate stay “hot” inside than outside?

Before attempting to answer this question, let’s do an experiment that will help us develop a model for this cooling situation. Use the temperature probe to collect data on a cup of hot (not boiling) water cooling to room temperature. Determine the room temperature.

1. Describe the data. (exponential growth or decay) Explain your choice.
2. Develop a function of the form $f(x) = ab^x + c$ (where c = the room temperature) to model the data.
3. What is the independent variable? The dependent variable?
4. What would constitute an appropriate domain and range?
5. Interpret the intercepts in terms of the cooling function.
6. Is the cooling function even, odd or neither?
7. Describe its behavior as x? ? .
8. Graph your function. How well does it fit the data?
9. Find the exp regression. Compare it to your function.
10. Newton’s Law of Cooling is

$$T(t) = T_m + (T_0 - T_m)e^{-kt}$$

for an appropriate value of k, where

$T(t)$ = the temperature of the object at time t (in minutes),

T_m = temperature of the surrounding medium,

T_0 = initial temperature of the object.

Use Newton's Law of Cooling to argue that it would be smart to stay inside for your "hot chocolate" break if your hot chocolate is 115? and k ? 0.254.