Unit 5

Inverse Functions: Logarithms, and Inverse Trigonometric Functions

Introduction

Inverse functions have the effect of "undoing" each other. For a function f and its inverse f^{-1}

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x$$

The domain of a function is the same as the range of its inverse and the range of the function is the same as the domain of the inverse. Their graphs are reflections of each other over the line y = x. In this unit we will investigate the inverses of some functions that we have studied in earlier units. We will look at logarithmic functions, which are inverses of the exponential functions, and we will consider the inverse trigonometric functions.

In order for the inverses of functions to also be functions, it is often necessary to restrict the domain of the original functions. Students often find the concept of restricting the domain of a function quite difficult. An activity has been included in this unit which will give them the opportunity to work through the process for themselves.

Problem 1[1]

How Much Do You Remember?

You were among a group of students who participated in a psychological experiment. You attended several lectures on technology advances in the 20th century. At the conclusion of the final lecture, you and the other students were given an exam on the material covered in the lectures. Each month for a year after the exam, you and your group were retested to see how much of the material you retained. The average scores for the group was given by the *human memory model*

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$$f(t) = 83 - 6\ln(t + 1)$$

where t is the time in months.

1. What was the original average score in your group?
2. How much did you and the others remember after 2 months? After 6 months? A year later?
3. What would constitute an appropriate domain and range for your memory function?
4. Interpret the intercepts. Will you live long enough to forget everything you learned at the lectures?
5. What is the behavior of the function as $t \to \infty$?

One Solution

1. The original score would be the score at time (t) = 0.

$$f(0) = 83 - 6\ln(0 + 1)$$
$$f(0) = 83 - 6(0)$$
$$f(0) = 83$$

2. After 2 months,

$$f(2) = 83 - 6\ln(2 + 1)$$
$$f(2) = 76$$

After 6 months,

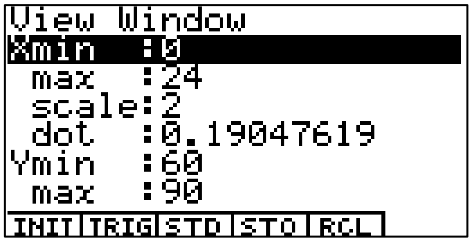
$$f(6) = 83 - 6\ln(6 + 1)$$
$$f(6) = 71$$

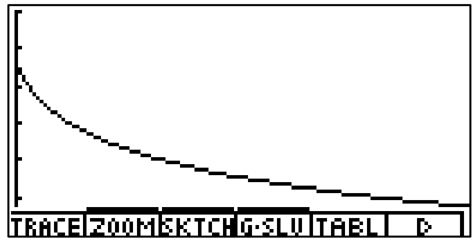
After 12 months,

$$f(12) = 83 - 6\ln(12 + 1)$$
$$f(12) = 68$$

3. In GRPH-TBL it will be necessary to use *x* and *y* instead of *t* and *f(t)*. Substituting *x* for *t* and *y* for *f(t)*, the domain of the model would be $\{x \mid x \geq 0\}$. If you were around 18 when you first took the test, it would not make sense to consider a domain larger than

about 1000. The range of the model would be $\{y \mid y \le 83\}$, however, it would not make sense to consider that one would remember a negative amount of information.



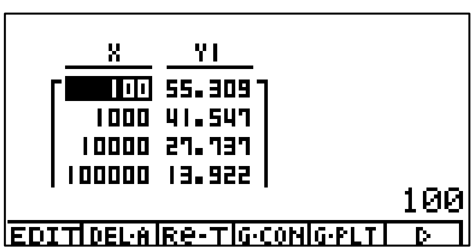


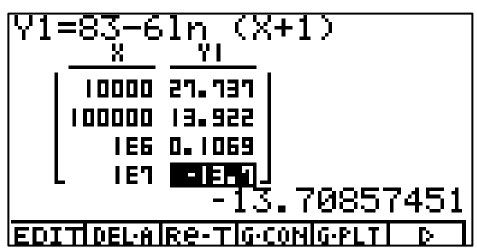
4. The y-intercept is 83, the value of the function at t = 0. The x-intercept, or the zero of the function is

$$83 - 6\ln(x+1) = 0$$
$$83 = 6\ln(x+1)$$
$$\frac{83}{6} = \ln(x+1)$$
$$e^{\frac{83}{6}} = x+1$$
$$e^{\frac{83}{6}} - 1 = x$$
$$x \approx 1017982$$

It is unlikely that one would live over 8,000 years.

5. As $x \to \infty$ the value of the function continues to decline. As the months go by, we remember less and less.





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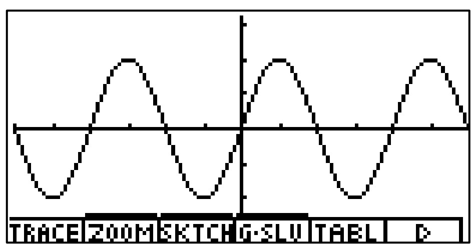
Problem 2[2]

The *Vertical Line Test* states that a relation is a function if and only if each vertical line in the coordinate plane intersects the graph of the equation in at most one point. If a vertical line is reflected over the line y = x, the image is a horizontal line. In order for a function *f* to have an inverse f^{-1} that is a function, *f* must pass the *Horizontal Line Test*, that is to say that the graph of *f* cannot intersect any horizontal line more that once.

Graph y = sin x on your calculator. How well does it do on the *Horizontal Line Test*?



In order for *f*(x) = sin x to have an inverse function, the domain must be restricted so that for each y there is one and only one x.

Your mission:

1. Decide on a restricted domain that would meet the following generally accepted criteria:
   - ?? The domain should include the angles between 0 and $\frac{?}{2}$ because these are the measures of the acute angles in a right triangle.
   - ?? On the restricted domain, the function should take on all values in the range exactly once.
   - ?? If possible, the function should be continuous on the restricted domain.
2. Sketch a graph of y = sin x with this restricted domain.
3. Reflect the graph over the line y = x to obtain a graph of $y = \sin^{-1} x$.

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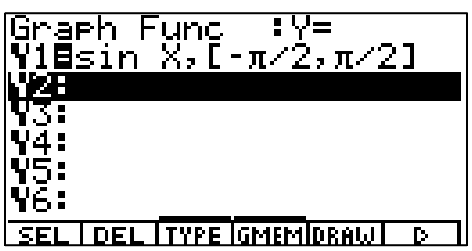
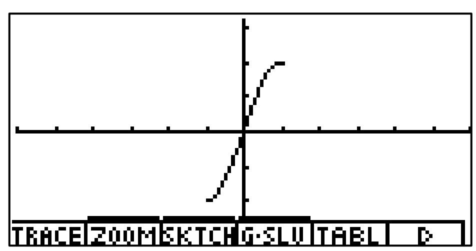
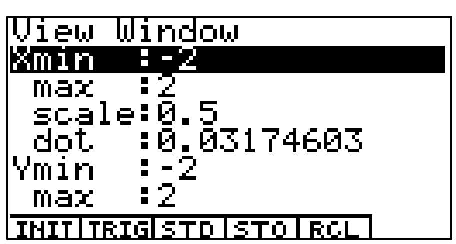
4. Give the coordinates of at least four points on the graph of y = $\sin^{-1}$ x.
5. Graph y = $\sin^{-1}$ x on your calculator. Compare this graph with your sketch in (3). Explain any differences.
6. Repeat this procedure with the cosine function. Did you choose a different domain. Why or why not?
7. Repeat this procedure with the tangent function.

One Solution

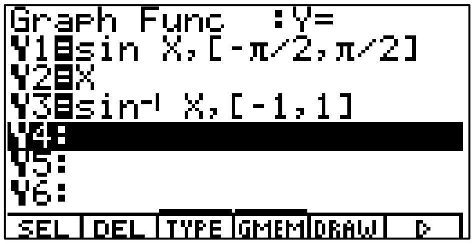
1. If we look at the graph of $f(x) ? \sin x$, we note that all the values in the range are between –1 and 1 so we must produce a domain that would include all these values *exactly once* to satisfy the second criterion. According to the first criterion, we must find a domain that would include the angles between 0 and $\frac{?}{2}$. These points in the domain would correspond to points in the range between 0 and 1. Next, we must add the points to the domain that would correspond to values in the range between –1 and 0. If we select the angles between $-\frac{?}{2}$ and 0, our domain will take on all the values in the range exactly once, thereby satisfying the second criterion. Further, the function is continuous over this restricted domain and satisfies the third and final criterion.
2. Student sketches.

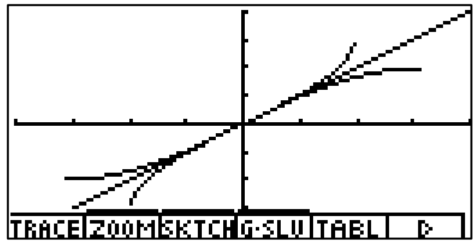






3. Student sketches.





4. Answers will vary. Examples: (0,0), (1, $\frac{?}{2}$), (-1, $\frac{?}{2}$), (.5, $\frac{?}{6}$)
5. Answers will vary. Presumably, the graph and the sketch will be similar.
6. If we look at the graph of $f(x) ? \cos x$, we note that all the values in the range are between –1 and 1 so we must produce a domain that would include all these values *exactly once* to satisfy the second criterion. According to the first criterion, we must find a domain that would include the angles between 0 and $\frac{?}{2}$. These points in the domain would correspond to points in the range between 0 and 1. Next we must add points to the domain which would correspond to values in the range between 0 and $?1$. If we select the angles between $\frac{?}{2}$ and $?$, our function will take on all the values in the range exactly once, thereby satisfying the second criterion. Further, the function is continuous over this restricted domain and satisfies the third and final criterion.
7. If we look at the graph of $f(x) ? \tan x$, we note that the values in the range include all real numbers, so we must produce a domain that would include all these values *exactly once* to satisfy the second criterion. According to the first criterion, we must find a domain that would include the angles between 0 and $\frac{?}{2}$. These values in the domain would give us range values from 0 to $?$. Next we must add points to the domain which would correspond to values in the range between $? ?$ and 0. If we select the angles between $?\frac{?}{2}$ and 0, our domain will take on all the values in the

range exactly once, thereby satisfying the second criterion. Further, the function is continuous over this restricted domain and satisfies the third and final criterion.

Problem 3

Artists and museum curators agree that one's perception of a painting is a function of the angle at which he or she views the painting. Suppose you are sitting on the bench in the middle of the large room in the Louvre Museum in Paris. The famous *Mona Lisa* is hanging on the wall in front of you. The bottom of the painting is 30 cm above your eye level and the painting is 117 cm from top to bottom. Your viewing angle is formed by the line of vision to the bottom and top of the painting.

1. If the bench you are sitting on is x meters from the painting, express your viewing angle ? as a function of that distance.
2. Graph the viewing angle function and trace to find the maximum value of ? .
3. If the bench is 15 m from the painting, what can you conclude about the position of the bench?

Problem 4[3]

The minimum required ventilation rate in terms of the air space per student in a public school classroom can be modeled by the function

$$f(x) = 80.4 - 11 \ln x, \qquad 100 \ ? \ x \ ? \ 1500$$

In the model, x is the air space per student in cubic feet and f(x) is the ventilation rate in cubic feet per minute.

1. Graph the function and trace to approximate the required ventilation rate if there are 300 cubic feet of air space per student.
2. A classroom is designed for 30 students. The air-conditioning system in the room has the capacity of moving 450 cubic feet per minute. Determine the ventilation rate per child, assuming that the room is filled to capacity.
3. Use the graph in #2 to estimate the air space per child.
4. Determine the minimum number of square feet of floor space required for the room if the ceiling height is 30 feet.

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[1] Larson, R. & Hostetler, R. (1997). PRECALCULUS. Boston: Houghton Mifflin Company.
[2] UCSMP. (1998). PRECALCULUS AND DISCRETE MATHEMATICS. Glenview, IL: Scott Foresman Addison Wesley.
[3] Larson, R. & Hostetler, R. (1997). PRECALCULUS. Boston: Houghton Mifflin Company.