

Unit 6

Continuity as a Property of Functions

Introduction

When an equation is used to describe how a body moves through space, it is a continuous function. There are many examples of numerical relationships in the world around us that can be represented by continuous functions. However, not all movement is continuous. One such example is the vibration of atoms in a hydrogen molecule. They can oscillate only at discrete energy levels and so cannot be described by a continuous function. Since discrete functions also play an important role in other areas such as computer science and statistics, it is of value to study the concept of continuity.

Problem 1¹

Discuss the continuity of the function $f(x) = \frac{x^2 - 2x + 8}{x^2 - 4}$ by answering the following questions.

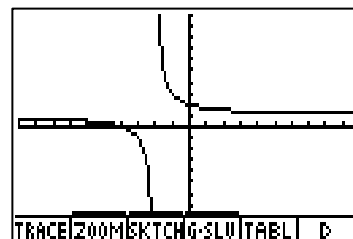
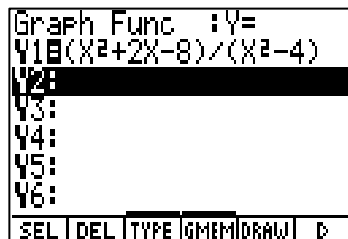
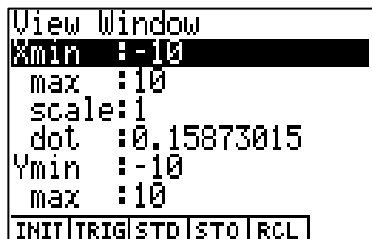
1. Graph the function to make a “guess” about points where the function might not be continuous.
2. Use the table feature to numerically explore the behavior of the graph around the point of discontinuity that you found in (1). What kind of discontinuity is this?
3. Is this the only point of discontinuity for the graph? How else can we discover other points of discontinuity?
4. Look at the graph again. Does there “appear” to be a point of discontinuity at this new value you found in (3)? Use the trace feature to examine the behavior of the function at this value for x . What does the calculator report?
5. Use the table feature to numerically explore the behavior of $f(x)$ for x close to the number you found in (4). What kind of discontinuity is this?
6. Verify analytically your findings about the points of discontinuity for the given function using limits.
7. State the intervals for which the function is continuous.
8. Can you redefine the function so that it is continuous everywhere? Can you redefine the function so that it is continuous for all x ?

9. Use the Intermediate Value Theorem to show that the function has a zero on the interval $[-6, -3]$.
10. What conclusions or general observations can you draw about the function and its graph?

One Solution

1. Graph the function to make a “guess” about points where the function might not be continuous.

This means we want to examine the graph of the function to look for places where there are “interruptions”. Holes, jumps, or gaps would be evidence of such breaks in a graph. We need to see a complete graph to make sure that we don’t miss any points at which the function is not continuous. Use the Casio Algebra FX 2.0 calculator to graph the function. Choose the GRPH-TBL Mode from the MAIN MENU. Enter the function in Y1 and set the viewing window (SHIFT OPTN). If it is not clear how to set the viewing window, we can start with the standard viewing window (STD found at the bottom of the screen in the View Window). Then ESC from the View Window and DRAW.

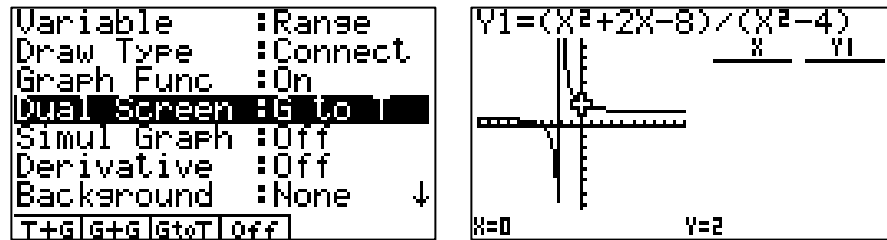


It appears that there is a break in the graph, called a point of discontinuity, near $x = 2$.

2. Use the table feature to numerically explore the behavior of the graph around the point of discontinuity that you found in (1). What kind of discontinuity is this?

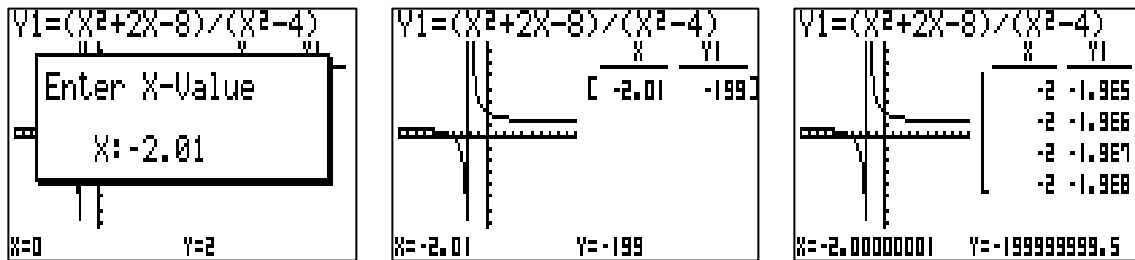
We will use the Graph to Table feature of the FX 2.0 calculator to examine the behavior of the function around $x = 2$. Select SET UP (CTRL F3) and turn the

Dual Screen on to “G to T” (F3). Now redraw the graph and activate TRACE.

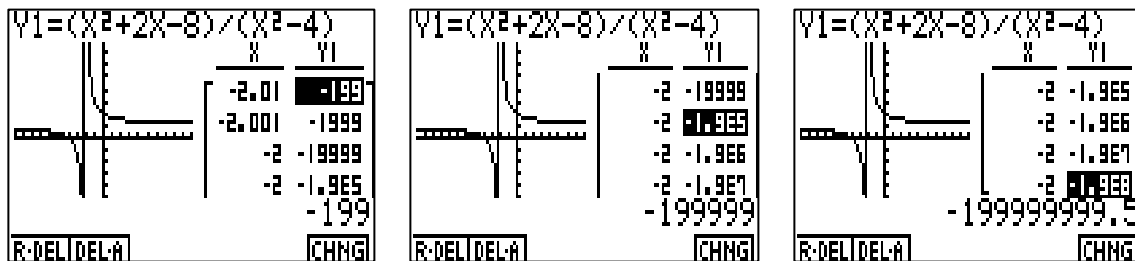


We will repeatedly use the left arrow to move the pointer and press EXE to store the coordinates in the number table. First watch what happens to the values of $f(x)$ as x is approaching -2 from the left. Choose some numbers to substitute in for x that are very close to -2 on the left side, i.e.

$x = -2.01, -2.001, -2.0001, -2.00001, -2.000001, -2.0000001, -2.00000001$.



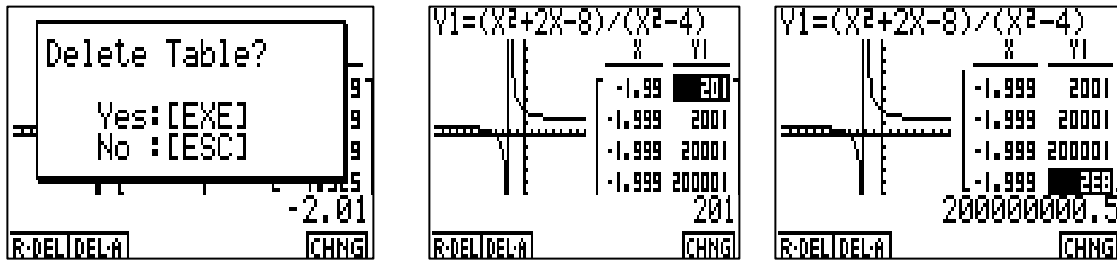
To study the numbers in the table more carefully we can make the table rather than the graph the active screen. To do this press ESC and then CHNG (F6). Then use the up arrow to move the cursor to the top of the table and then the right arrow to move the cursor to the Y1 column. Scroll down through the function values as x gets closer and closer to -2 from the left. Observe that the function appears to be decreasing without bound.



Now let's do the same thing to see what happens to the values of $f(x)$ as x is approaching -2 from the right. Choose some numbers to substitute in for x that are very close to -2 on the right side, i.e.

$x = 1.99, 1.999, 1.9999, 1.99999, 1.999999, 1.9999999, 1.99999999$.

First we need to delete all the values in the table. Select DEL-A. Then redraw the graph, activate TRACE, and store the above values in the table. Again make the table the active screen and scroll through the function values in the table.



Observe that the function values seem to be increasing without bound as x approaches -2 from the right. This is called an infinite discontinuity (vertical asymptote).

- Is this the only point of discontinuity for the graph? How else can we discover other points of discontinuity?

Look for other discontinuities by determining the domain of the function. The

function $f(x) = \frac{x^2 + 2x + 8}{x^2 - 4}$ is a rational function. The domain will be the set of all

real numbers excluding those numbers for which the denominator is zero. Factor the denominator and observe that $x = 2, -2$. Thus the domain is the set of all real numbers excluding $x = 2, -2$. We have already considered the behavior of the graph at $x = -2$ and can clearly see that the function is not continuous at that point. There is also a point of discontinuity at $x = 2$ which we did not notice when we looked at the graph.

- Look at the graph again. Does there “appear” to be a point of discontinuity at this new value you found in (3)? Use the trace feature to examine the behavior of the function at this value for x . What does the calculator report?

Under SET UP turn off the graph to table feature. Reset the View Window to standard and re-graph the function. Using the trace feature, at $x = 2$ the calculator gives an error message. Although there does not appear to be a point of discontinuity when we look at the graph, the calculator alerts us to this break in the curve since it is not able to evaluate the function at $x = 2$.

- Use the table feature to numerically explore the behavior of $f(x)$ for x close to the number you found in (4). What kind of discontinuity is this?

This time we will use the Table and Graph feature of the FX 2.0 calculator. Under SET UP turn the Dual Screen on to T+G. To specify the table range, in the GRPH-TBL mode, select RANG. We want to examine the behavior of the function when x is close to 2. One possibility is to start at 1.999, end at 2.001, and increment in amounts of .0001. Let's also change the View Window to correspond. Look at the graph. It appears that the function lies between 0 and 2 for x close to 2. Use this information to set your View Window.

Variable	:Range
Draw Type	:Connect
Graph Func	:On
Dual Screen	:T+G
Simul Graph	:Off
Derivative	:Off
Background	:None
T+G G+G Gt&T off	

Table Range	
X	
Start	:1.999
End	:2.001
Pitch	:1E-04

View Window	
Xmin	:1.999
max	:2.001
scale	:1E-04
dot	:3.2258E-05
Ymin	:0
max	:2
INIT TRIG STD STO RCL	

Now graph the function and turn on the graph link, G-Link (F6 then F4). Scrolling down the Y1 column of function values, the pointer jumps to the corresponding point on the graph. It appears as x approaches 2 from the left, that the function is approaching 1.5 although it is undefined at $x = 2$.

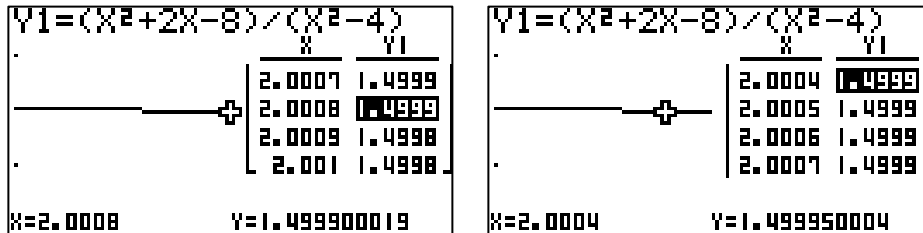
Y1=(X ² +2X-8)/(X ² -4)	
X	Y1
1.999	1.5001
1.9991	1.5001
1.9992	1.5001
1.9993	1.5
X=1.999	Y=1.500125031

Y1=(X ² +2X-8)/(X ² -4)	
X	Y1
1.9993	1.5
1.9994	1.5
1.9995	1.5
1.9996	1.5
X=1.9996	Y=1.500050005

Y1=(X ² +2X-8)/(X ² -4)	
X	Y1
1.9997	1.5
1.9998	1.5
1.9999	1.5
2	ERROR
X=2	Y=ERROR

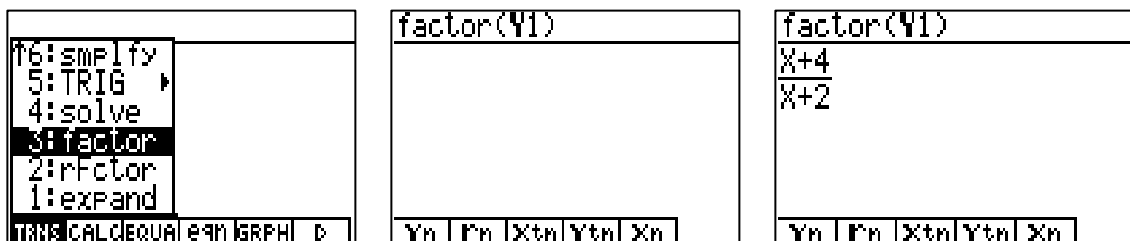
Now jump the cursor down to the bottom of the table and move up the table of function values to simulate moving towards $x = 2$ from the right. It also appears as x approaches 2 from the right, that the function is approaching 1.5 although it is undefined at $x = 2$.

This is called a removable discontinuity at $x = 2$.



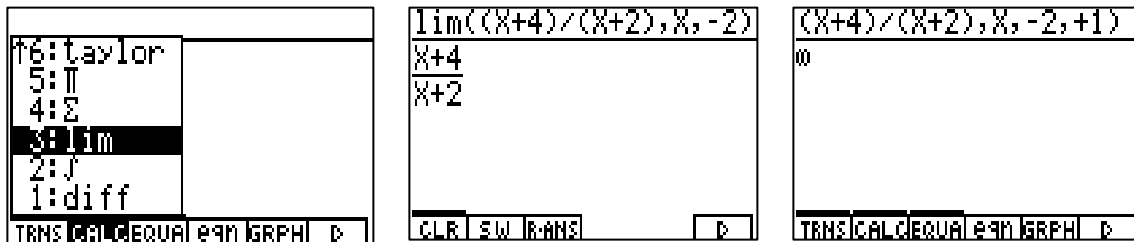
6. Verify analytically your findings about the points of discontinuity for the given function using limits.

A rational function has a vertical asymptote for those points c at which the numerator is not 0 but the denominator is zero. We will begin by simplifying the function using the FX 2.0 calculator. From the Main Menu choose the CAS menu. Clear all equations (F6, CLR). Then choose TRNS and factor and enter the function. Note that it is already stored in Y1, so we simply select VARS, Yn and press 1. Then EXE.



Thus, $f(x) = \frac{x^2 + 2x + 8}{x^2 + 4} = \frac{x + 4}{x + 2}$ for $x \neq 2$. The graphs of the two functions coincide at all values of x other than $x = 2$ and there is a vertical asymptote at $x = -2$. Using the CAS menu we will evaluate the limits of the function both at

$x \neq 2$ and at $x = 2$. Choose \lim under the CALC menu and recall (R-ANS) our last answer, $\frac{x + 4}{x - 2}$ to the input line. Find the limit of the function from both sides as $x \rightarrow 2$. Use +1 (-1) to specify the limit is from the right (left) side. The limit from the right is ∞ (undefined) and $-\infty$ (undefined) from the left. Thus the result is undefined which corroborates our observation that there is a vertical asymptote at $x = 2$.



Now evaluate the limit as $x \rightarrow 2$ to corroborate our determination that the function is approaching 1.5 from both sides at $x = 2$.



7. State the intervals for which the function is continuous.

The function is continuous on the intervals $(-\infty, 2)$, $(2, 2)$, and $(2, \infty)$.

8. Can you redefine the function so that it is continuous everywhere? Can you redefine the function so that it is continuous for all $x \neq 2$?

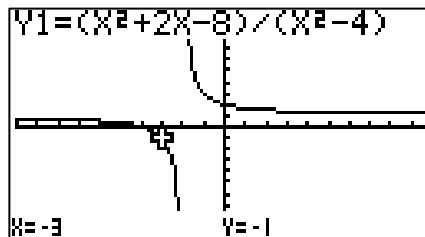
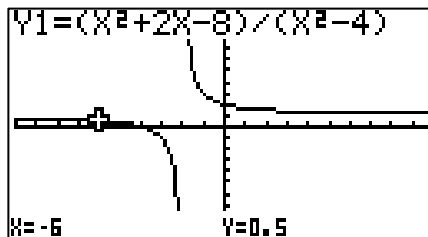
The function cannot be redefined so that it is everywhere continuous because it has an infinite discontinuity at $x = 2$. However, since there is a removable discontinuity at $x = 2$ the function is continuous for all $x \neq 2$ when it is defined as follows:

$$f(x) = \frac{x^2 - 2x + 8}{x^2 - 4} \text{ for } x \neq 2 \text{ and } f(x) = 1.5 \text{ for } x = 2.$$

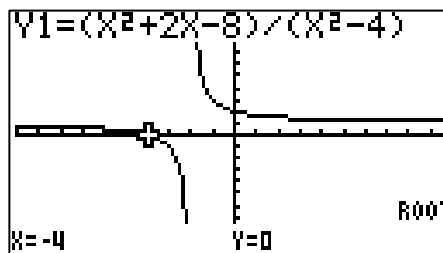
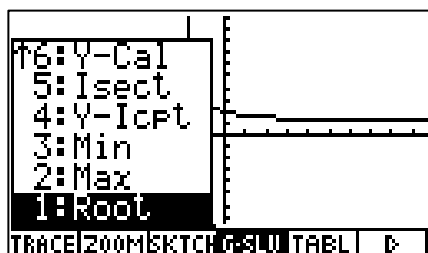
9. Use the Intermediate Value Theorem to show that the function has a zero on the interval $[-6, -3]$.

The Intermediate Value Theorem says that if a function is continuous on a closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c)=k$.

Turn off the Dual Screen and re-graph the function. Use the trace key to find the values of the function at $x = -6$ and at $x = -3$.



Observe that $f(-6) > 0$ and $f(-3) < 0$. Applying the Intermediate Value Theorem we can conclude that there must be some c in $[-6, -3]$ such that $f(c) = 0$, that is c is a zero for the function on that interval. Use the calculator to find the zero. Select G-SLV and then ROOT in the graphing window to find that there is a root at $x = -4$.



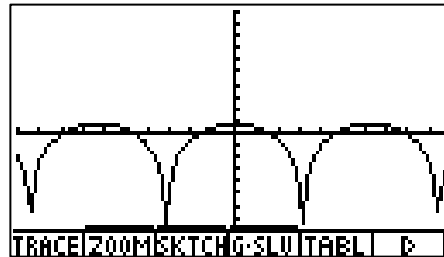
10. What conclusions or general observations can you draw about the function and its graph?

Discussion

Problem 2²

Find the intervals for which the function $f(x) = \ln(1 + \cos x)$ is continuous.

One Solution



It is not clear by examination of the graph that there are any points of discontinuity. We know that the functions $h(x) = \ln x$ and $g(x) = 1 + \cos x$ are continuous. There is a theorem that states that the composition of two functions is continuous everywhere it is defined. The function $f(x) = \ln(1 + \cos x)$ is defined when $1 + \cos x > 0$, i.e. it is not defined when $\cos x = -1$. However, $-1 < \cos x < 1$, so $f(x)$ is undefined when $\cos x = -1$. The solutions to $\cos x = -1$ are $x = \pi, 3\pi, 5\pi, \dots$. We can conclude that $f(x) = \ln(1 + \cos x)$ is discontinuous at odd multiples of π and continuous on the intervals in between. To verify using limits, evaluate the limits

$\lim_{x \rightarrow \pi^-} \ln(1 + \cos x)$ and $\lim_{x \rightarrow \pi^+} \ln(1 + \cos x)$ to find that there is a vertical asymptote at all odd multiples of π .



Problem 3³

Long distance calls (direct-dial) between Atlanta and Detroit cost \$1.04 for the first 2 minutes and \$0.36 for each additional minute or fraction thereof. Describe the cost of a call in terms of the time t (in minutes) using the greatest integer function. Determine the continuity of the function by examining the graph and using limits.

Problem 4⁴

Use the Intermediate Value Theorem to show that there is a solution to the equation $\ln x = e^{-x}$ in the interval (1,2) and find the solution.

¹ Larson, R., Hostetler, R., & Edwards, B., (1998), Calculus of a Single Variable. Boston: Houghton Mifflin Company.

² Stewart, J., (1999), Calculus: Early Transcendentals. California: Brooks/Cole Publishing Company.

³ Larson, R., Hostetler, R., & Edwards, B., (1998), Calculus of a Single Variable. Boston: Houghton Mifflin Company.

⁴ Stewart, J., (1999), Calculus: Early Transcendentals. California: Brooks/Cole Publishing Company.