Unit 7

Concept of the Derivative and the Derivative at a Point

Introduction

Finding the slope of the graph of a function at a given point is a fundamental geometric problem that can be solved using calculus. The slope of the graph at a point is defined as the slope of the tangent line to the graph at that point. The slope of the tangent line at a point can be approximated using the slope of a secant line through the point of tangency and a second point on the curve. Thus, as the second point is chosen closer and closer to the point of tangency, the slope of the tangent line is defined at the limit of the slopes of the secant lines.

Problem 1[1]

Give a geometric, numerical, and analytical analysis of the derivative of the function $f(x) = x^2$ by discussing the following:

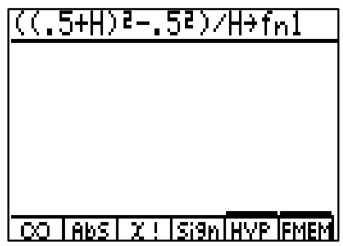
1. Use the difference quotient to estimate the slope of the function at $x = .5$ by constructing a table of secant line slopes for the secant lines between the point of tangency and the points $x = 1, .75, .55, .51, .501, .5001$.
2. In the same viewing window, graph the function and the secant lines.
3. Use the limit of the difference quotient to exactly find the slope of the graph at $x = .5$. Use the calculator as an aid in this process and do it by hand to compare the results.
4. Local linearity is a fundamental property of differentiable functions. Use the dual graph feature of the Casio, Algebra FX 2.0 calculator to zoom in on the graph of $f(x)$ at $x = .5$.
5. Write the equation to the tangent line at $x = .5$.
6. Use the difference quotient with $h = .001$ to estimate $f'(x)$ at $x = 0, 1, 2, 3, 4, 5$. Then guess at the equation for $f'(x)$.
7. Find $f'(x)$ exactly using the limit of the difference quotient.
8. Discuss the continuity of the function both at $x = .5$ and over the entire domain.

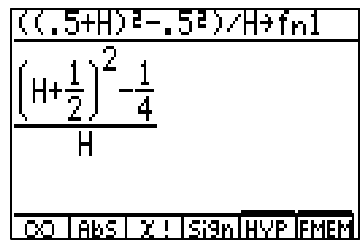
One Solution

1. Use the difference quotient to estimate the slope of the function at $x ? .5$ by constructing a table of secant line slopes for the secant lines between the point of tangency and the points $x ? 1, .75, .55, .51, .501, .5001$.

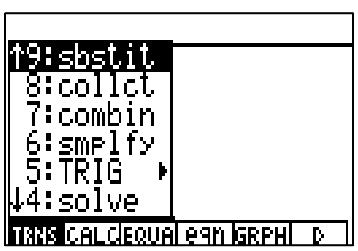
   The slope of a secant line through the point of tangency, $x ? .5$, and a second point is

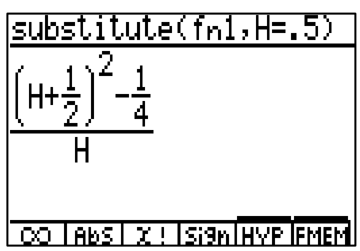
   $m_{\text{sec}} ? \dfrac{f(.5 ? h) ? f(.5)}{h} ? \dfrac{(.5 ? h)^2 ? .5^2}{h}$, where h is the change in x. We will find slopes of successive secant lines using the CAS Mode of the FX 2.0 calculator. Clear all equations if necessary (F6-CLR). Store the right-hand side of the above equation as a function. The fn notation is found under OPTN/FMEM.





Press AC/ON to clear the input line. Select TRNS/sbstit to substitute in successive values for h. In the same manner, generate the slopes of the secant lines to complete the table below.



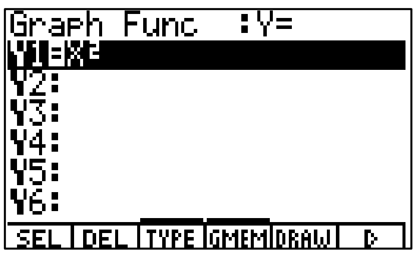


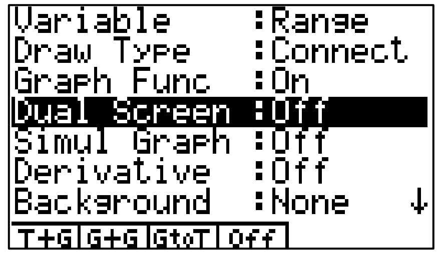
Close point to the right	h	slope
1	.5	3/2
.75	.25	5/4
.55	.05	21/20
.51	.01	101/100
.501	.001	1001/1000
.5001	.0001	10001/10000

As the second point gets closer and closer to the point of tangency, $x \to .5$, the slope of the secant line appears to be approaching the value of 1.

2. In the same viewing window, graph the function and the secant lines.

Choose the GRPH-TBL Mode from the Main Menu. Enter the function $f(x) = x^2$ in Y1. Choose CTRL/SET UP (F3) to turn the dual screen off. Return to the CAS Mode.

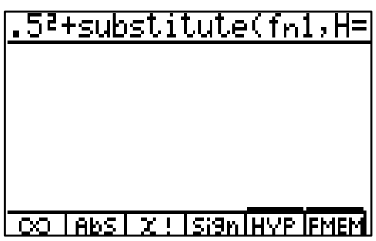


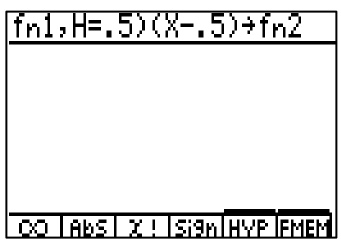


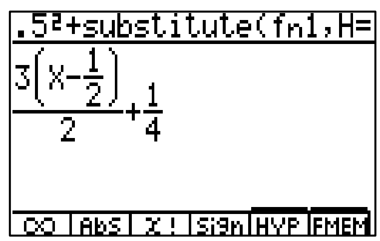
To graph the secant lines for the decreasing values of h, we'll use the stored function for the secant line slope. The general equation for each secant line will be

$$y = f(.5) + m_{\text{sec}}(x - .5) = .5^2 + m_{\text{sec}}(x - .5).$$

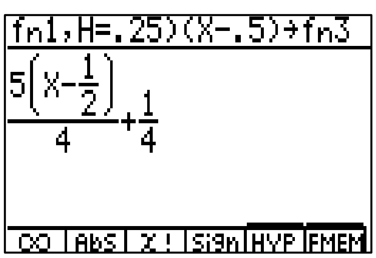
Construct and store a new function for the secant line when h= .5 by using the substitute command to generate the slope for the equation as follows.

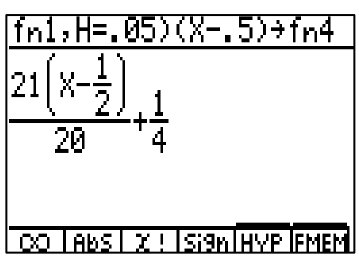


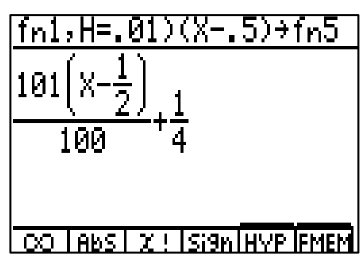




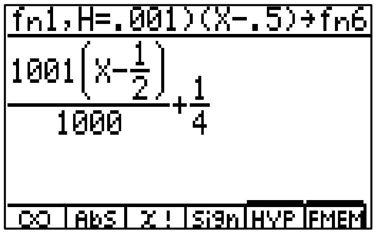
Successively change the value substituted for h as well as the function name (fn) to create all the secant line equations.

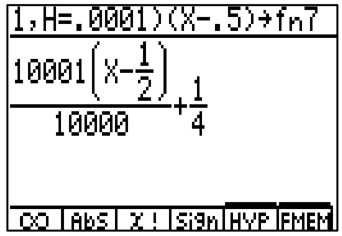


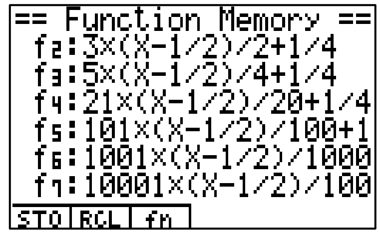




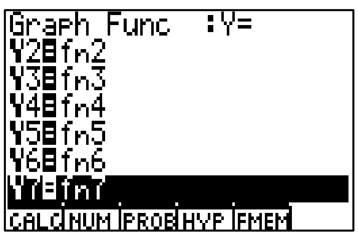
Finally, to check to see that all the secant lines are properly stored, choose OPTN, FMEM(F6) and SEE (followed by EXE). Pressing the down arrow will show functions 2-7 for the 6 secant lines we generated.

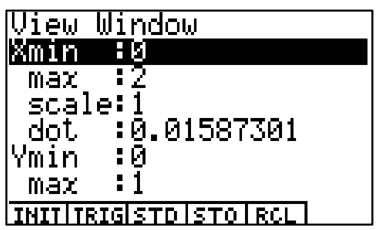


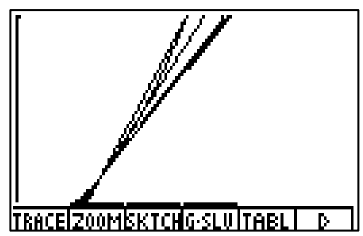




Now to graph the function and the secant lines, return to the GRPH-TBL Mode. In Y2 through Y7 enter the 6 secant lines stored as fn2-fn7. Change the View Window to get a closer look as the secant lines are drawn. One set of parameters might be as shown below. Then graph to see the secant lines approaching the tangent line with progressively smaller choices for h.

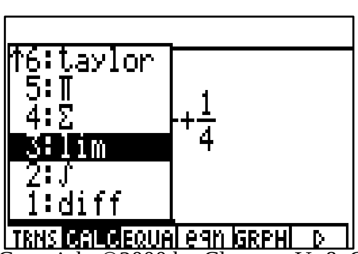


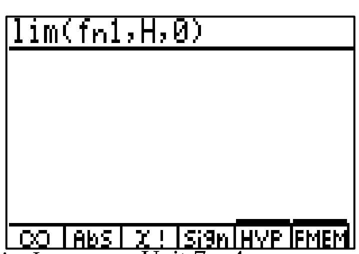




3. Use the limit of the difference quotient to exactly find the slope of the graph at $x = .5$. Use the calculator as an aid in this process and do it by hand to compare the results.

The exact slope of the graph at the point $x = .5$ can be found by evaluating the limit of the difference quotient as h goes to zero, i.e. $\lim_{h \to 0} \frac{(.5 + h)^2 - .5^2}{h}$. Return to the CAS Mode. Clear the input line. Recall that the difference quotient as defined above is stored in fn1. Select the CALC menu and choose lim.

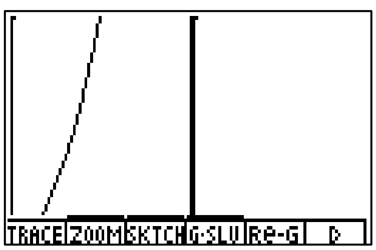


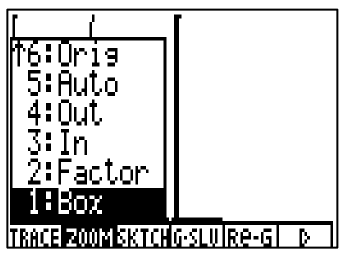


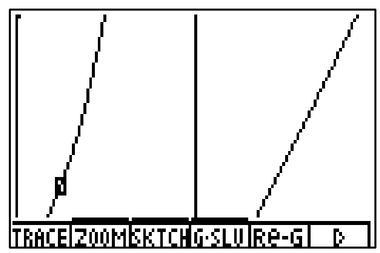
We see that the limit of the difference quotient is 1, thus the slope of the tangent line at $x ? .5$ is 1.

4. Local linearity is a fundamental property of differentiable functions. Use the dual graph feature of the Casio, Algebra FX 2.0 calculator to zoom in on the graph of $f(x)$ at $x ? .5$.

   The property of local linearity states that if we look closely near a point on a smooth curve, the curve will appear to be a line near that point. More important it will look like the tangent line at that point. Return to the GRPH-TBL Mode and delete all functions except for $Y1 ? x^2$. The viewing window may remain the same or you may wish to alter it. Graph the function and select ZOOM/BOX. Draw a box around the point where $x ? .5$ on the graph of the function to see the portion of the graph within the box on the sub-screen. Notice that it appears to be essentially a line.

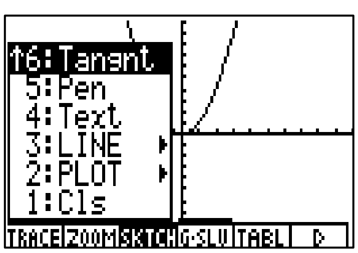


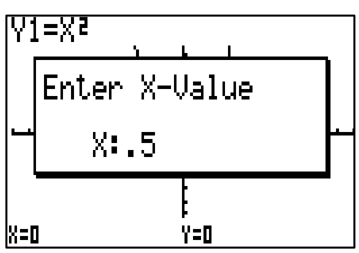


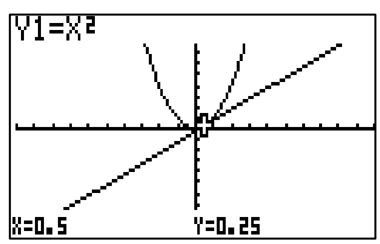


5. Write the equation to the tangent line at $x ? .5$.

   Turn off the the G+G feature of the Dual Screen and change the View Window back to the standard parameters. Regraph the function and select SKTCH/TANGNT to sketch the tangent line at $x ? .5$. Enter .5 as the X-Value and press EXE twice to see the tangent line.



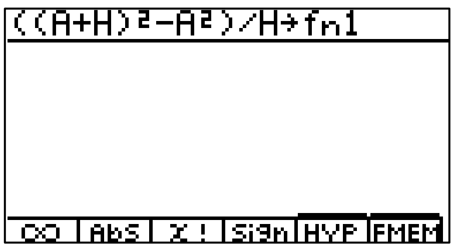


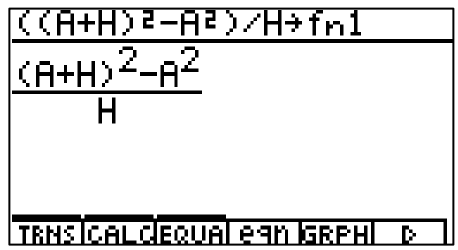


The equation for the tangent line at $x ? .5$ is $y ? f(.5) ? m_{\tan}(x ? .5) ? .25 ? 1(x ? .5)$.

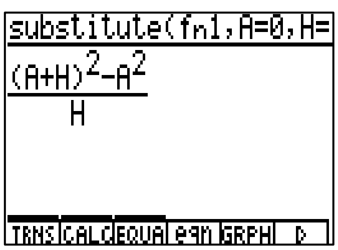
6. Use the difference quotient with $h ? .001$ to estimate $f'(x)$ at $x ? 0, 1, 2, 3, 4, 5$. Then guess at the equation for $f'(x)$.

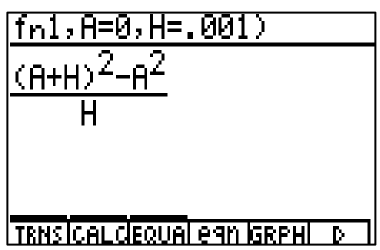
We will use the difference quotient, $\dfrac{f(a ? h) ? f(a)}{h} ? \dfrac{(a ? h)^2 ? a^2}{h}$ where a is a value for x in the domain of our function and h is the change in a to guess at an equation for the derivative of $f'(x)$. Delete all functions by selecting the SYSTEM Mode from the Main Menu. Choose F1 (Memory Usage), scroll down to Function Mem, and press DEL. Then return to the CAS Mode and clear all equations. Enter and store the difference quotient in Fn1.

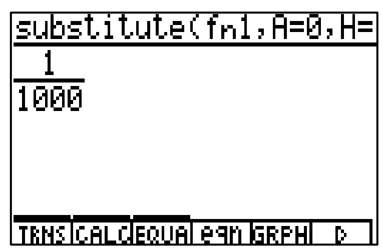




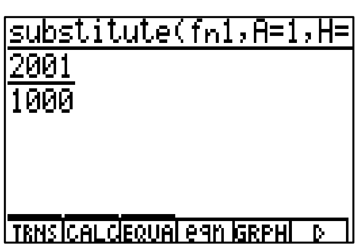
Choose sbstit from TRNS to substitute in values for a and h. For all choices of a we will use only one value for h, namely .001. Begin with setting $a ? 0$.

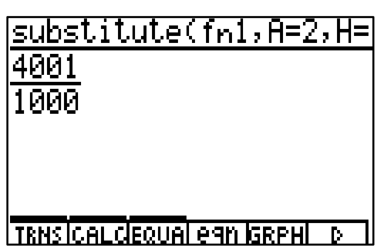


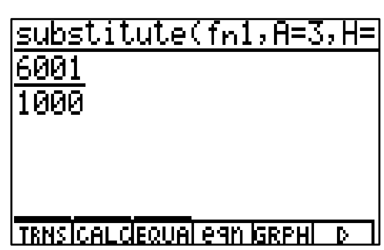




Our numerical guess of the derivative of the function at $x ? 0$ is $f'(0) ? .001$. Fill this value in the table and continue to find the remaining approximations.







Continuing in the same manner we fill in the values in the table. It appears that $f'(x)$ is approximately twice x.

x	$f'(x)$
0	.001
1	2.001
2	4.001
3	6.001
4	8.001
5	10.001

7. Find $f'(x)$ exactly using the limit of the difference quotient.

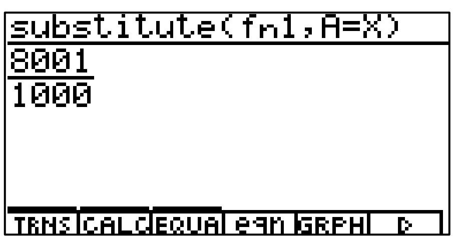
The definition of the derivative of a function is

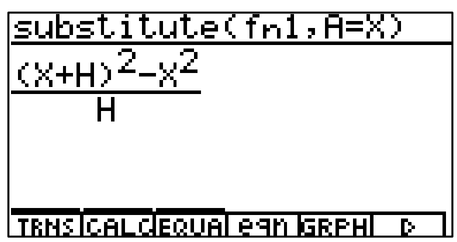
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ provided the limit exists.

For our function $f(x) = x^2$, to find the derivative we need to evaluate

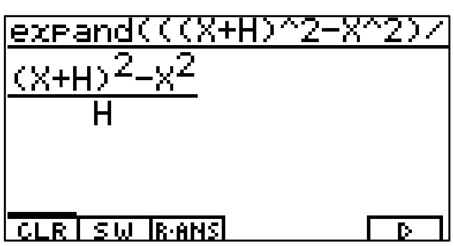
$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}.$$

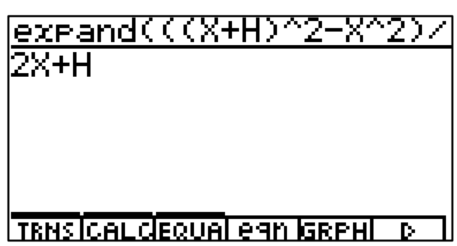
Since we already have the difference quotient stored as fn1, we can simply make the substitution of x for a with no value substituted in for h.



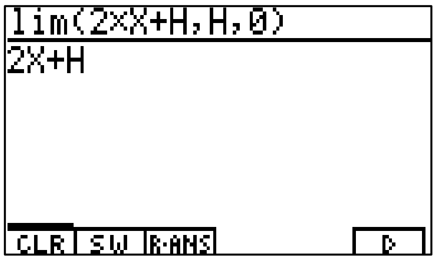


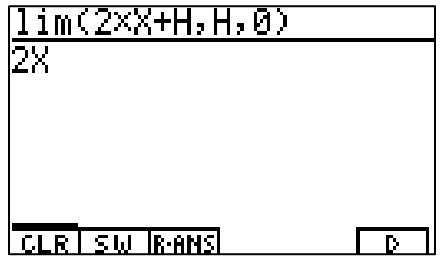
To see some intermediate steps in evaluating this limit, choose TRNS/expand (or smplfy) and then retrieve the above expression (F6 R-ANS) to simplify the expression.





Now evaluate the limit of this simplified expression by choosing CALC/lim. Again use R-ANS and then EXE to see that $f'(x)$? $2x$ as we guessed in the previous step.





8. Discuss the continuity of the function both at x ? .5 and over the entire domain.

   The function is a polynomial function and is everywhere continuous.

Problem 2[2]

The National Rifle Association experienced a decline in membership in the early 1990's. As the decade continued there appeared to be some renewed interest in the organization. Consider the data in the table below and answer the following questions if possible. Decide if the question should be answered using the data or if you should fit a continuous model to answer the question.

Year	1990	1991	1992	1993	1994	1995
NRA membership (in millions)	2.8	2.6	2.7	3.2	3.5	3.6

(The Associated Press, *Anderson Independent-Mail*, May 20, 1995, page A1.)

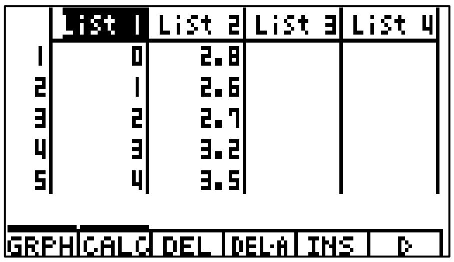
1. What was the NRA membership in 1993?
2. What was the average rate of change in the membership from 1992 through 1995?
3. What was the average rate of change in NRA membership during the first half of 1994?
4. How quickly was the NRA membership increasing in 1994?
5. When was NRA membership increasing most rapidly?

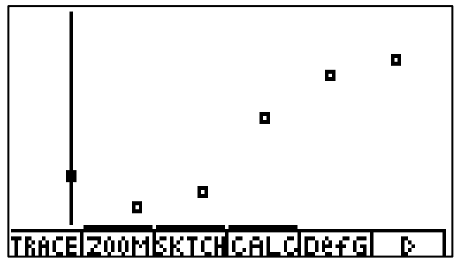
One Solution

1. What was the NRA membership in 1993?
   The data includes the actual 1993 membership, so it is not necessary to fit a model to the data to answer the question. The NRA membership in 1993 was 3.2 million.
2. What was the average rate of change in the membership from 1992 through 1995?
   Use the data in table to answer this question since exact values for the years 1992 and 1995 are given. The average rate of change in the membership from 1992 through 1995 is $\frac{3.6 ? 2.7}{3} ? .3$ million.

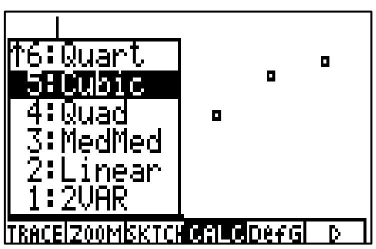
3. What was the average rate of change in NRA membership during the first half of 1994?

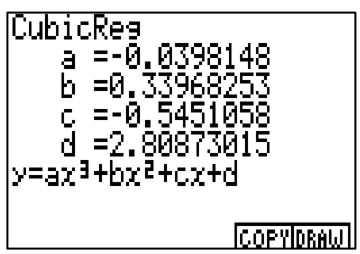
Growth of membership is not continuous and we are not given the membership in mid-year. So, for purpose of estimating we will use the FX 2.0 calculator to generate a continuous model. Let t=0 represent the year 1990. Plot the data to decide what type model to use. A cubic model appears to be a good choice for a model.

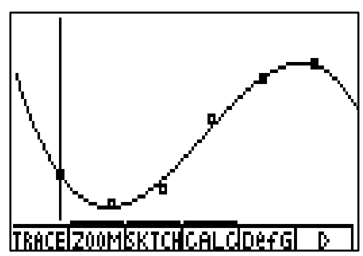




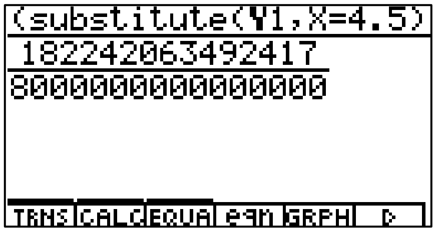
In the graph screen choose CALC/Cubic and EXE to generate the coefficients for the model. Then press COPY and EXE to copy and save the regression coefficients to the GRPH-TBL Mode formula area. Now draw the function with the data points to examine the fit.

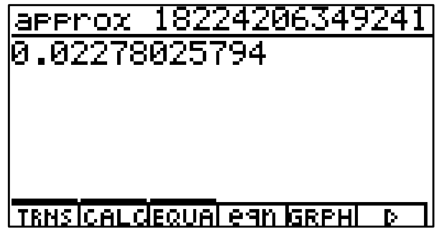






The cubic model fit to the data is $M(t) = -0.0398t^3 + 0.3397t^2 - 0.5451t + 2.8087$ million members t years after 1990. We can calculate the average rate of change during the last half of 1994 by computing $\frac{M(4.5) - M(4)}{4}$. Since the cubic model is stored in Y1, we can go to the CAS Mode to compute the average rate of change.

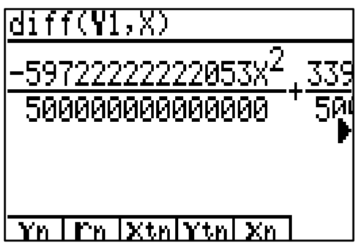


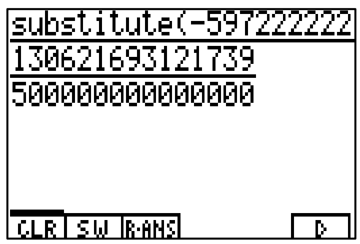


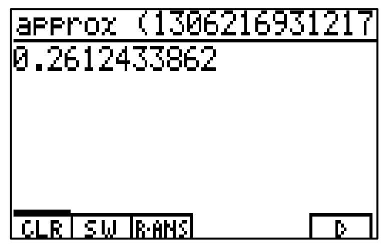
Therefore, the average rate of change during the last half of 1994 was approximately 0.02 million members.

4. How quickly was the NRA membership increasing in 1994?

   The continuous model is required to answer this question since it involves instantaneous rate of change. Again using the CAS Mode, we find the derivative of the cubic model and substitute in t=4 to find the instantaneous rate of change in 1994.



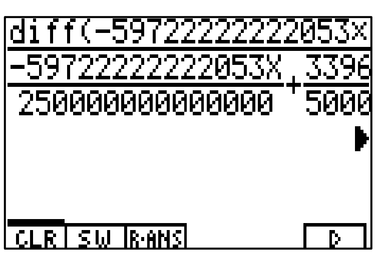


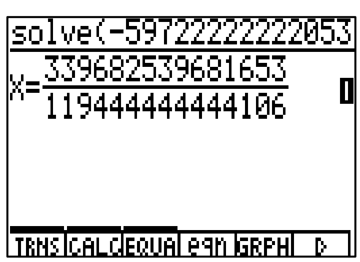


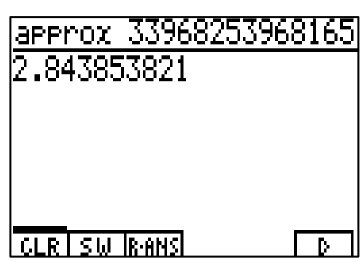
In 1994 the NRA membership was increasing approximately at a rate of .26 million per year.

5. When was NRA membership increasing most rapidly?

   A function is increasing most rapidly where its derivative has a maximum value. This is where the second derivative is zero, i.e. an inflection point. Find the second derivative of the membership function and solve (TRNS/solve) to find the inflection point. The NRA membership was increasing most rapidly in late 1992.







Problem 3[3]

Consider the data given below for the amount of fish (in millions of pounds) produced for human food by fisheries in the Unites States. Discuss the relationship between continuity and differentiability by discussing the following:

Year	Amount of fish (millions of pounds)
1970	2537
1972	2435
1975	2465
1977	2952
1980	3654
1982	3285
1985	3294
1987	3946
1990	7041

(Statistical Abstract, 1994.)

1. Align the data so that t=0 in 1970, t=2 in 1972, etc. Look at the scatter plot and then find a piecewise model for the data by dividing the data in 1980. Graph the model with the data points (as a scatter plot).
2. Is the function defined by the model continuous? Where should we definitely check for continuity?
3. Does the derivative of the piecewise model exist in the year 1980? Use the principle of local linearity and zoom in close to the point at t=10.
4. Could we estimate the instantaneous rate of change in 1980 using the model we generated? Does this mean there is no instantaneous rate of change in the production of fish in 1980?

Problem 4[4]

Find the linearization of the function $f(x) ? e^{?2x}$ at a =0 and use it to approximate the number $e^{0.03}$. For what values of x is the linear approximation accurate to within 0.5? Estimate this interval in the following manner. The linear approximation should lie between the curves obtained by shifting the curve $f(x) ? e^{?2x}$ upward and downward by the amount 0.5 directions. (Justify this analytically.) Graph the function, the two functions representing the shifts you obtain, and the tangent line. Use the ZOOM and TRACE features of the FX 2.0 calculator to aid you in finding an interval for x, i.e. find the x-coordinates for the points at which the tangent line is no longer trapped between the two shifted graphs of the function. (Round to be safe.)

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[1] TICAP (1995), TICAP: Technology Intensive Calculus for Advanced Placement, Clemson, SC: Clemson University.

[2] LaTorre, D., Kenelly, J., Fetta, I., Carpenter, L., & Harris, C., (1998), Calculus Concepts: An Informal Approach to The Mathematics of Change, Boston: Houghton Mifflin Company.

[3] LaTorre, D., Kenelly, J., Fetta, I., Carpenter, L., & Harris, C., (1998), Calculus Concepts: An Informal Approach to The Mathematics of Change, Boston: Houghton Mifflin Company.

[4] Stewart, J., (1999), Calculus: Early Transcendentals. California: Brooks/Cole Publishing Company.