

Unit 9

Applications of Derivatives

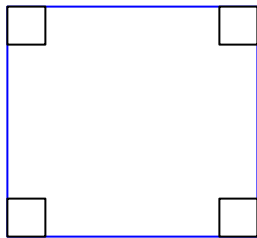
Introduction

How often do we want to determine the least cost, the least amount of time, the smallest size, the largest profit, the greatest strength, or the longest distance?

Determining a maximum or minimum value is one of the most important common applications of calculus. A problem-solving strategy for optimization problems should include numerical, graphical, and analytical analysis. The first step is to write an equation that describes the quantity being optimized. Calculus techniques are then used to find the optimal value. Finally, the results are summarized.

Problem 1¹

Your mother wants to send a care package of candy mints to your sister at college. There are no empty boxes in the house. However, in the garage she finds a square piece of cardboard, 24 inches on each side. A box can be made (without a top) by cutting equal squares from the corners and turning up the sides. She wants the box to hold as much as possible. Find out what size square should be cut from each corner so that the box formed will have the largest volume.



We will investigate this problem from three perspectives – numerically, graphically, and analytically – by answering the following questions.

1. What is the smallest sized square that can be removed from the corner? What is the largest sized square that can be removed from the corner? Construct a table of possible volumes for squares of length 0-12 inches (use whole numbers) removed from the corner. Use the volumes in the table to make a guess at what size square needs to be removed to give the maximum volume.

2. Give a function that describes the quantity that is to be maximized. What is the domain of this function, i.e. what are the values of x for which the stated problem makes sense?
3. Graph the function on the given interval and look for the maximum volume.
4. Use the function and calculus techniques to find the size of the square we must remove to have a box with the maximum volume.
5. At each step, summarize your results.

One Solution

1. The length of the side of the smallest square that can be removed must be larger than zero. Since a square is removed from each corner, the side of a square must be no more than one half the length of a side of the cardboard, i.e. $\frac{1}{2}(24)=12$ inches.

Complete the table below to make a guess at what size square needs to be removed from the corner to give the maximum volume.

Using the the Casio Algebra FX 2.0 calculator to perform the calculations, select the RUN-MAT Mode from the MAIN MENU. For example, to calculate the volume when the height is 1 inch, you would have

The image shows a calculator screen in MAT mode. The display shows the expression $1(24-2(1))^2$ and the result 484 . The 'MAT' indicator is visible in the bottom left corner.

Height	Length/Width	Volume
0 inches	$24 - 2(0)$	$0[24 - 2(0)]^2 = 0$ cubic in
1 inch	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$ cubic in
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$
3	$24 - 2(3)$	$3[24 - 2(3)]^2 = 972$
4	$24 - 2(4)$	$4[24 - 2(4)]^2 = 1024$ *****
5	$24 - 2(5)$	$5[24 - 2(5)]^2 = 980$
6	$24 - 2(6)$	$6[24 - 2(6)]^2 = 864$
7	$24 - 2(7)$	$7[24 - 2(7)]^2 = 700$
8	$24 - 2(8)$	$8[24 - 2(8)]^2 = 512$
9	$24 - 2(9)$	$9[24 - 2(9)]^2 = 324$
10	$24 - 2(10)$	$10[24 - 2(10)]^2 = 160$
11	$24 - 2(11)$	$11[24 - 2(11)]^2 = 44$
12	$24 - 2(12)$	$12[24 - 2(12)]^2 = 0$

To Summarize: Based on the values for volumes in the table, we estimate the maximum volume to be 1024 cubic inches when a square with a side length of 4 inches is removed from each corner of the cardboard.

2. Give the function that describes the quantity that is to be maximized.

$$\begin{aligned} \text{Volume} &= (\text{height}) \times (\text{base}) \\ &= (\text{height}) \times (\text{length}) \times (\text{width}) \end{aligned}$$

In our problem, the base is a square, so the length equals the width.

To write the function, define the variable x to be the length of the side of the square.

Use a numerical example from the table as a guideline,

$$3[24-2(3)]^2 = 972 \text{ cubic inches.}$$

Then the volume is defined to be

$$V(x) = x(24 - 2x)^2$$

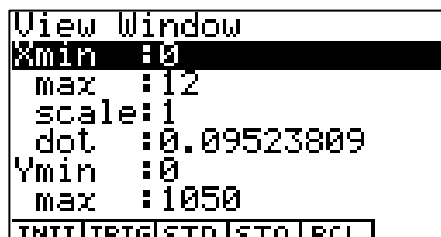
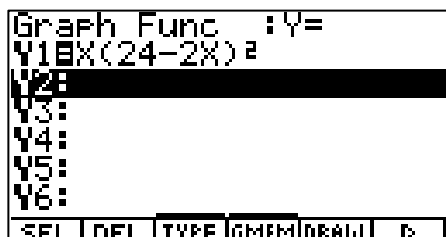
What is the domain of the function? Based on our discussion in question 1, the size of the square must be between 0 and 12, $0 \leq x \leq 12$.

To Summarize: The function that describes the maximum volume for a box formed from a square piece of cardboard with sides of length 24 inches is

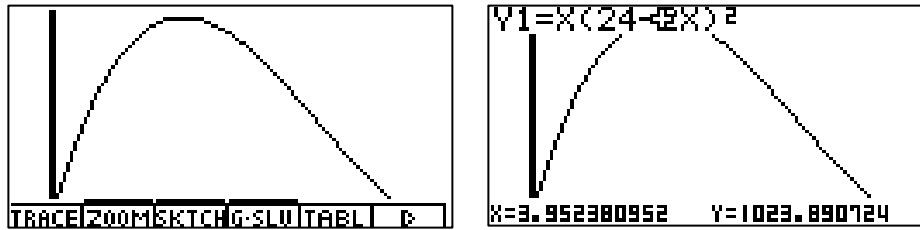
$$V(x) = x(24 - 2x)^2 \quad \text{with} \quad 0 \leq x \leq 12.$$

3. Graph the volume function on the given domain and look for the maximum volume.

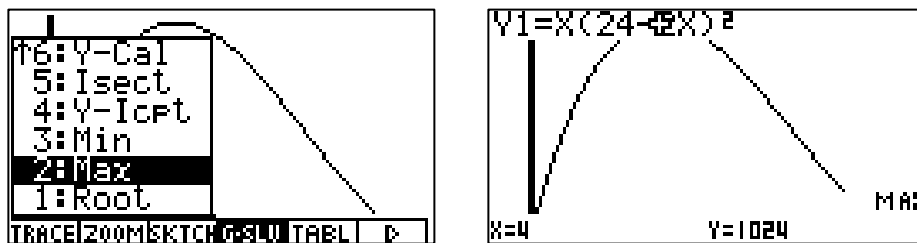
Using the Casio Algebra FX 2.0 calculator to graph the function, select the GRPH-TBL Mode from the MAIN MENU. Enter the function in Y1 and set the viewing window (SHIFT OPTN). Choose a View Window that gives a complete graph of the function on the given interval. Using the table you we constructed in question 1, observe the minimum volume is 44 cubic inches and the maximum volume is 1024 cubic inches. So one possible choice for the viewing window is $X_{\min} = 0$, $X_{\max} = 12$, $Y_{\min} = 0$, and $Y_{\max} = 1050$.



Now DRAW the graph. Use the TRACE key to approximate the maximum volume.

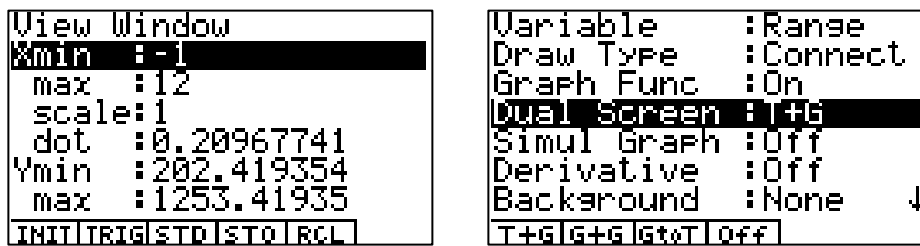


The Casio Algebra FX 2.0 calculator provides a number of different methods for analyzing functions and graphs. Now let's use the solver capabilities to improve our approximation of the maximum volume. Use the G-SLV menu under the graph and select 2:Max (maximum value).



To Summarize: The maximum volume of 1024 cubic units is obtained when a square of 4 inches is cut from each corner of the square cardboard.

Further Exploration: Let's use the table and graph combination for the Casio Algebra FX 2.0 calculator and expand our chosen domain for the function, say $-1 \leq x \leq 13$. Change the View Window accordingly. On the SET UP screen (CTRL F3) turn the Dual Screen on to T+G.



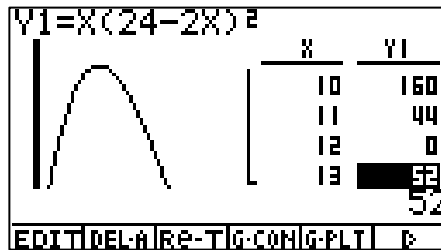
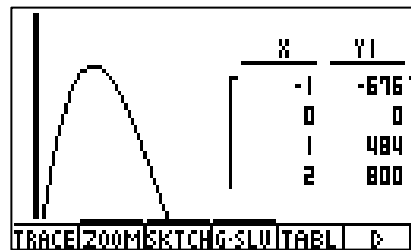
To specify the table range, in the GRPH-TBL mode, select RANG (F6 then F2).

To generate a table of values as the variable x ranges from -1 to 13 , set Start equal to -1 , End equal to 13 , and pitch equal to 1 . (Pitch specifies the incremental value for the variable x .)

Table+Graph :Y=	
Y1:	$X(24-2X)^2$
Y2:	
Y3:	
Y4:	
Y5:	
Y6:	
SEL RANG G-VAR G-PLT TABEL D	

Table Range	
X	
Start:	-1
End :	13
Pitch:	1

Now DRAW the function. Use the table feature (F5 TBL) to explore function values for our domain $-1 \leq x \leq 13$.



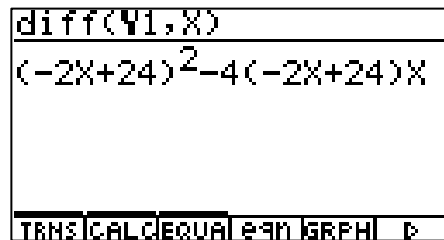
Observe that when $x = 1$ that $y = 676$. Clearly this is not a viable solution for our problem. However, when $x = 13$ then $y = 52$. We have already determined that we can not make a box when a square of length 13 inches is cut from the piece of cardboard. Why does the function seem to be increasing in value again? Analytical work involving the derivative of the function will explain the behavior of the function for $x > 12$.

- Use the function in 2 and calculus techniques to find the size of the square we must remove from each corner to have a box with the maximum volume.

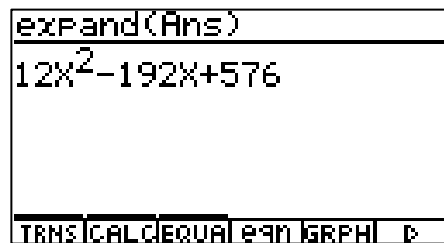
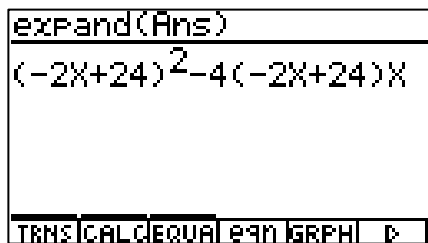
Recall that relative extrema occur at critical numbers of the function. To find the critical numbers, we first differentiate the function with respect to x .

Use the CAS (Computer Algebra System) Mode of the Casio Algebra FX 2.0 calculator to differentiate the volume function. Since the function is already assigned

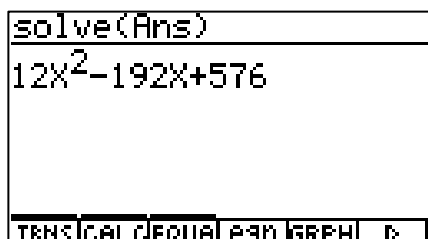
to graph memory Y1, we can recall this function to differentiate. Under the CALC key choose diff and in the input area enter the function to be differentiated, i.e. $\text{diff}(Y1,X)$. (Note: Y_n is found under VARS.)



Again we use the CAS feature to simplify the derivative. Under TRNS (formula transformation menu) choose expand and in the input area enter $\text{expand}(\text{Ans})$.



So we have found that the first derivative of the volume function is $V'(x) = 12x^2 - 192x + 576$. Critical numbers can be found by setting the first derivative equal to 0 and solving. Thus we must solve the equation $0 = 12x^2 - 192x + 576$. Under TRNS choose solve and in the input area enter $\text{solve}(\text{Ans})$ to find the roots of the equation.



An alternative way to find the roots is to use the CAS feature to factor the equation. Use the up arrow key to return to the previous screen. Under TRNS choose factor and in the input area enter factor(Ans).

```
factor(Ans)
12X2-192X+576
-----
TRNS|CALC|EQW|PRN|GRPH|D|
```

```
factor(Ans)
12(X-12)(X-4)
-----
TRNS|CALC|EQW|PRN|GRPH|D|
```

Using the calculator we find the roots of the equation and thus the critical numbers are

$$x = 4 \text{ and } x = 12.$$

The critical number $x = 12$ is not in the domain of our function. It makes no sense to cut corners out that have a length of 12 inches. Thus, we only need to look at the critical number $x = 4$ to determine if it is a relative maximum or relative minimum. We do this by using either the First or Second Derivative Test. To use the Second Derivative Test, we will use the CAS mode to calculate the second derivative of the volume function. Use the up arrow key to return to the screen with the first derivative. Under CALC choose diff and in the input area enter diff(Ans).

```
diff(Ans)
24X-192
-----
TRNS|CALC|EQW|PRN|GRPH|D|
```

We have found that the second derivative of the volume function is $V''(x) = 24x - 192$. The function has a relative maximum at the critical number $x = 4$ if the function is concave down at that point, that is the second derivative

is negative. To evaluate $V'(4)$, under TRNS choose sbstiti (substitute) and in the input area enter substitute(Ans, $x = 4$) to find that $V'(4) = -96 < 0$.

substitute(Ans,X=4)
24X-192
TRNS CAL CEQA EQN GRPH D

substitute(Ans,X=4)
-96
TRNS CAL CEQA EQN GRPH D

Thus $V(x)$ is concave down when $x = 4$ and there is a relative maximum at $x = 4$. To find the minimum value of the function, under TRNS choose sbstiti (substitute) and in the input area enter substitute(Y1, $x = 4$) to find that $V(4) = 1024$.

substitute(Y1,X=4)
1024
Yn Fn Xtn Ytn Xn

To Summarize: The maximum volume of 1024 cubic inches is found when squares with sides of length 4 inches are removed from the corners of the cardboard.

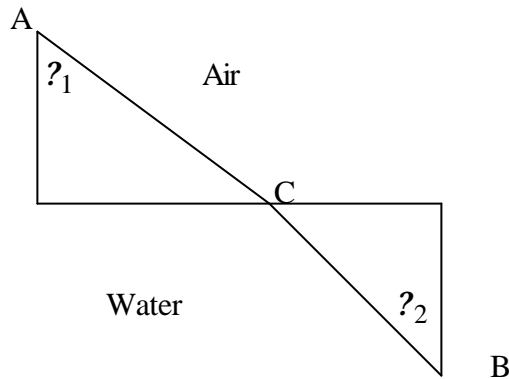
Further Exploration: . Why does the function seem to be increasing in value again at $x = 13$? Recall that we discovered that there is also a critical number at $x = 12$. Again using the Second Derivative Test, we find that $V'(12) = 96 > 0$. Thus $V(x)$ is concave up when $x = 12$, and there is a relative minimum at $x = 12$. To the right of $x = 12$, the function, $V(x)$, is increasing.

- Using numerical, graphical, and analytical analysis we have determined that to form a topless box with maximum volume, a square with side length of 4 inches would have to be cut from each corner. The maximum volume would then be 1024 cubic units.

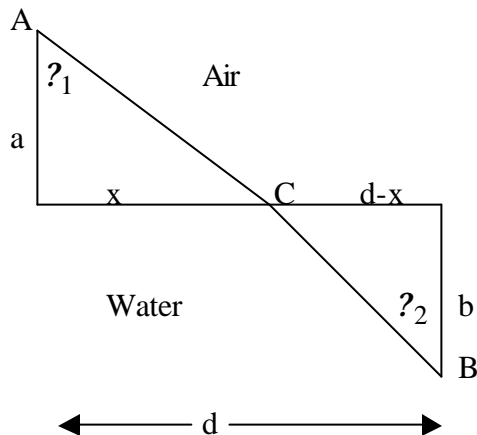
Problem 2²

Fermat's Principle states that a ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimizes the time taken. Let V be the velocity of light in the air and W the velocity of light in the water. Let θ_1 (the angle of incidence) and θ_2 (the angle of refraction) be as shown in the picture below. Show that

$\frac{\sin \theta_1}{\sin \theta_2} = \frac{V}{W}$, which is known as Snell's Law.

**One Solution**

First we need to define a function that gives the total time for the ray of light to travel from point A in the air to point B in the water following a path ACB. Define the distances as shown in the picture below.



Since time=distance/rate, let the total time be defined by

$$T(x) = (\text{the time from A to C}) + (\text{the time from C to B})$$

$$= \frac{\sqrt{a^2 + x^2}}{V} + \frac{\sqrt{b^2 + (d-x)^2}}{W}, 0 \leq x \leq d.$$

To find the minimum distance we need to take the derivative with respect to x . Using the FX 2.0 calculator, we find that the derivative is

$$T'(x) = \frac{x}{V\sqrt{x^2 + a^2}} + \frac{x-d}{W\sqrt{b^2 + (x-d)^2}}.$$

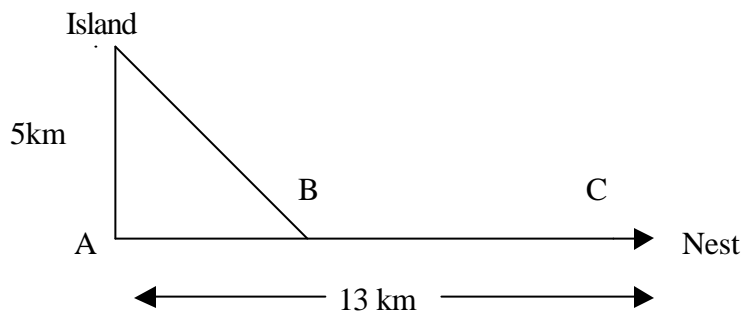
Observe that $T'(x) = \frac{\sin \theta_1}{V} + \frac{\sin \theta_2}{W}$. Since the minimum occurs when $T'(x) = 0$, we

have that $\frac{\sin \theta_1}{V} = -\frac{\sin \theta_2}{W}$. Thus $\frac{\sin \theta_1}{\sin \theta_2} = -\frac{V}{W}$.

Problem 3³

Ornithologists have determined that during daylight hours there are some species of birds that try to avoid flights over large bodies of water. Because air generally rises over land and falls over water during the day, they believe that the birds require more energy to fly over water during the day. Suppose there is an island 5 km from the nearest point A on a straight shoreline. A bird, from the species avoiding flights over large bodies of water during the day, is released from the island and flies to a point B on the shoreline. From there it flies to its nesting area C on the shoreline. We will assume that the bird instinctively chooses a path to minimize its energy expenditure. Let points A and C be 13 km apart.

- Suppose it takes 1.4 times as much energy for a bird to fly over water as land. If a bird wants to minimize total energy expended traveling from the island to its nesting area, to what point B on the shoreline should the bird fly?
- Let W be the energy (in joules) per kilometer the bird requires to fly over water and L be the energy (in joules) per kilometer to fly over land. Consider the ratio of energy required to fly over water to energy required to fly over land, W/L . What would a large value of the ratio versus a small value of the ratio mean relative to the bird's flight? Find the ratio, W/L , that corresponds to the minimum expenditure of energy.
- Find the value of the ratio, W/L , if the bird
 - flies directly from the island to the nesting area?
 - flies directly to the point A on the shoreline and then along the shoreline to the nest?
- Suppose an ornithologist finds that the bird reaches the shore 4 km from point A. How much more energy did it take for the bird to fly over the water than the land?



Problem 4⁴

A ladder 25 feet long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 feet per second. Discuss the following.

1. How fast is the top moving down the wall when the base of the ladder is 10 feet and 20 feet from the wall?
2. Draw the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 10 feet from the wall.
3. Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 10 feet from the wall.

¹ Larson, R., Hostetler, R., & Edwards, B., (1998), Calculus of a Single Variable. Boston: Houghton Mifflin Company.

² Stewart, J., (1999), Calculus: Early Transcendentals. California: Brooks/Cole Publishing Company.

³ Stewart, J., (1999), Calculus: Early Transcendentals. California: Brooks/Cole Publishing Company.

⁴ Larson, R., Hostetler, R., & Edwards, B., (1998), Calculus of a Single Variable. Boston: Houghton Mifflin Company.