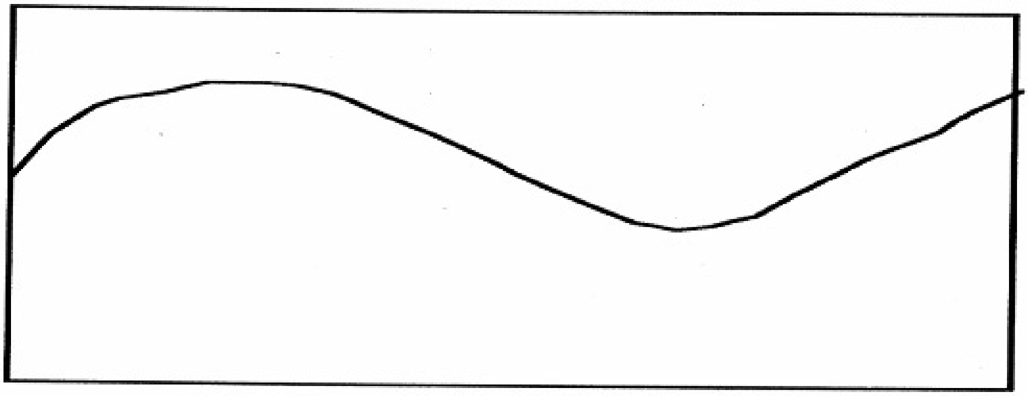
Unit 14

Numerical Approximations to Definite Integrals

Introduction

Hand held computing technology has moved the focus in calculus instruction away from closed form solutions of integrals to an emphasis on numerical solutions. Much has been said about the fact that many of the integrals that one encounters in real world problems do *not* permit close form solutions and numerical solutions are the only practical answer. *Close form solutions* to integrals refers to situations where you can find the antiderivative of the integrand, i.e., $\int f(x)\,dx = F(x) + C$ where $F'(x) = f(x)$ and $\int_a^b f(x)\,dx = F(b) - F(a)$. Obviously there are many instances where a function f(x) does not readily yield a function F(x) where $F'(x) = f(x)$. In many cases it is simply hard to find such an expression F(x) and, in other cases, there are functions where it is impossible to find an analytic expression that yields the necessary connection $F'(x) = f(x)$. For example $\int e^{-x^2} dx$ is an example where we know that no analytic expression exists that has e^{-x^2} as its derivative. In this section, we need to improve our understanding of the numerical methods that computing devices use to find the numerical values of definite integrals.

Computer algorithms that give numerical evaluations to definite integrals typically use limiting values of approximating rectangles very much like you studied in the introductory sections on integrals in your first calculus course. Lets take the example of water existing as a wave in a transparent rectangular aquarium.



Lets suppose that an instant photograph was taken of the water wave while it was in motion and we wanted to know the height of the water after it "settled down." Obviously there is an underlying function f(x) that describes the shape of the wave, but since we only have a picture of the water, we clearly will never know its analytic expression. And even more the case, we will never know an analytic expression that will yield under differentiation, the functional expression that describes the wave. What most people would do in this instance is to measure the depth of the water at several places and average the values. This is exactly the process that we were using in our first introduction to calculus where we used approximating rectangles to the area under the curve. We might not have seen it as "adding the values and dividing by N" but that was in fact the process that was taking place. In particular, we talked about a little strip of area under the curve (the approximating rectangle) and we added up all the approximating strips and divided by N (the number of rectangles). In particular, the expression we used is given by:

$$\int_a^b f(x)dx = \frac{(b-a)}{N}\sum_{i=1}^{N} f(x_i)\Delta x$$

Clearly we are adding up the rectangles and dividing by N to determine the height of the "average" rectangle. Then we multiply that average value by the length of the base interval to get an approximation of the area. Typically most calculus books explore several methods on inscribing the rectangles and the exploring of these alternatives is the basis of Problem 1.

Problem 1

In this problem, we look at the relationships between the approximating value of a definite integral derived when we use (1) approximating rectangles determined by the function's value at the right end point of the intervals, (2) approximating rectangles determined by the function's value at the left end point of the intervals, (3) approximations determined by inscribed trapezoids above the intervals, and (4) approximating rectangles determined by the function's value at the mid point of the intervals. There may be some results that surprise you.

First we investigate the approximating of a definite integral by using the values of the function at the left end points of the intervals, i.e., the left end point rule. To keep the arithmetic simple, lets take as our example the integration of f(x) = 1/x and the interval of integration is [1, 2]. We know that the astute reader will exclaim that the integrand, 1/x, has an antiderivative, i.e., ln |x| and that the value of the definite integral is simply $\ln(2) - \ln(1) = 0.6931471806$, but realize that this is an instructional example and we want to contrast precisely the different methods in oder to have insights into results that can be determined with only numerical integration. The steps of the analysis follow and you should work out each on your FX 2.0 calculator.

$$\int \frac{1}{x}dx = \ln|x| \qquad \text{thus} \qquad \int_1^2 \frac{1}{x}dx = \ln 2 - \ln 1 = 0.6931471806$$

Assume that we divide the interval [1,2] into n equal part of width 1/n and we approximate the value of the integral by adding up the small rectangles formed by the function's height above points on the interval times the width of the small partition intervals. Note that the small intervals have horizontal axes coordinates of

{1, 1+1/n, 1+2/n, 1+3/n, ... ,1+(n-1)/n, 2}. For the numerical approximation formed by rectangles at the right end points of the intervals we have:

$$\int_1^2 \frac{1}{x}dx \approx \frac{1}{n} * \sum_{i=1}^{n} \frac{1}{1+i(\frac{1}{n})} \quad \textit{Right End Point Rule}$$

For the numerical approximation formed by rectangles at the left end points of the intervals we have

$$\int_1^2 \frac{1}{x}dx \approx \frac{1}{n} * \sum_{i=0}^{n-1} \frac{1}{1+i(\frac{1}{n})} \quad \textit{Left End Point Rule}$$

For the numerical approximation formed by trapezoids using both the right and the left end points of the intervals we have

$$\int_1^2 \frac{1}{x}dx \approx \frac{1}{n} * \left(\frac{1}{2} + \sum_{i=1}^{n} \frac{1}{1+i(\frac{1}{n})} + \frac{1}{4} \right) \quad \textit{Trapezoid Rule}$$

For the numerical approximation formed by rectangles at the right end points of the intervals we have

$$\int_1^2 \frac{1}{x}dx \approx \frac{1}{n} * \sum_{i=0}^{n} \frac{1}{1+\frac{1}{2n}+i(\frac{1}{n})} \quad \textit{Midpoint Point Rule}$$

Notice that the right end point, the left end point and the trapezoid rule all have imbedded a basic sum, i.e.,

$$\sum_{i=1}^{n-1} \frac{1}{1+i(\frac{1}{n})} = f_n 1$$

If we call that common expression, fn1, then we see that the first three rules reduce to

$$\textit{Right End Point} = \left(f_n 1 + \frac{1}{2}\right)\frac{1}{n}$$

$$\textit{Left End Point} = \left(f_n 1 + 1\right)\frac{1}{n}$$

$$\textit{Trapezoid Rule} = \left(f_n 1 + \frac{1}{2} + \frac{1}{4}\right)\frac{1}{n}$$

We now take advantage of that connection to estimate the value of the definite integral with 10, 20, 50 and 100 divisions of the interval. A summary table of the values is:

subdivisions	Right End Pt	Left End Pt.	Trapezoid	Midpoint	Fn1
10	0.66877	0.71877	0.6937	0.69284	6.187714032
20	0.68080	0.70580	0.699330	0.69307	13.11606764
50	0.68817	0.69817	0.69317	0.69313	33.90860897
100	0.69065	0.69565	0.69315	Ma error	68.56534305

Since this example also has a known exact value, we can examine the errors present in each of the above approximations.

subdivisions	Right End Pt	Left End Pt.	Trapezoid	Midpoint
10	-0.12438	0.02562	0.00062	-0.00031
20	-0.01235	0.01265	0.00015	-0.00008
50	-0.00498	0.00502	0.00002	-0.00002
100	-0.00250	0.00250	0.00000	Ma error

This positions us to make a remarkable observation; "the right and left end point rules have about the same errors but they are opposite in sign." You should verify this observation and give a geometric explanation for the similarity. We also can observe that for this (concave down) curve, the trapezoid rule overestimates yet it is two orders of magnitude better in its approximations. You should again try to explain the geometric reasons for this.

But we also can make an astonishing observation; "the midpoint rule is the best of all and its error is about a third of the trapezoid.

Upon noting the observations about the left and right end points rules being under and over estimates of the same order of magnitude, one would arrive at a natural adjustment that would take advantage of this and average the right rule answer and the left rule answer to get a better result. In fact this "better result" is just the trapezoid rule. In that spirit we observe that the trapezoid rule and the mid point rule give opposite results, i.e., one overestimates concave up curves and the other underestimates concave up curves. But in this case the error in the midpoint rule is about half the error in the trapezoid rule. Consequently, we would best improve our answer by taking a weighted average of the two values. That is, twice the mid point result plus once times the trapezoid result all divided by three. This weighted average gives the very well known Simpson Rule that is usually described as one derived by fitting parabola's through the function values.

Problem 2

Consider a quarter of a unit circle in the first quadrant, i.e., $y = \sqrt{1 - x^2}$ for x=0 to x=1. We know that the area is $\pi/4$. When we integrate the y function numerically from 0 to 1 with a plain version of Simpson's Rule, we encounter difficulty finding an accurate integral because of the vertical tangent at x=1. If we use sophisticated numerical methods like Gauss Kronrad, then these problems are overcome by the numerical techniques in this algorithm assigning some of its error controls to regions where the error needs controlling. The Casio 9850 graphing calculator allows the user to set the integration routine to a plain Simpson Rule or alternatively, Gauss Kronrad.

Use a Casio 9850 graphing calculator and do the integration in several states of division points and different setting on the Simpson/Gauss-Kronrad switch. Compare your results.

Problem 3

Most individuals mistakenly think that you can always improve the accuracy of numerical integration problems by increasing the number of subdivisions. Find some examples of functions and numerical integration regions where this is not the case.

Problem 4

On the FX 2.0 graphing calculator, set the viewing window at the integer setting. Graph $y = .1\, x^2$ from $x = 0$ to $x = 5$. Estimate the number of pixels under the curve and above the x-axis from $x = 0$ to $x = 5$.

Write the expression for a numerical integration using a right hand end point rule and use it to estimate the number of pixels under the curve.